Homework No. 11 (Spring 2023)

PHYS 520B: ELECTROMAGNETIC THEORY

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Due date: Thursday, 2023 Apr 20, 4.30pm

1. (**20 points.**) The magnetic field associated to radiation fields is given by

$$c\mathbf{B}(\mathbf{r},t) = -\hat{\mathbf{r}} \times \frac{\mu_0 c}{4\pi}\frac{1}{r}\int d^3r' \left\{\frac{1}{c}\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}',t')\right\}_{t'=t_r}, \tag{1}$$

where the contribution to the field comes at the retarded time

$$t_r = t - \frac{r}{c} + \hat{\mathbf{r}}\cdot\frac{\mathbf{r}'}{c}. \tag{2}$$

The associated electric field is given by

$$\mathbf{E}(\mathbf{r},t) = -\hat{\mathbf{r}} \times c\mathbf{B}(\mathbf{r},t), \tag{3}$$

and satisfies

$$c\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r},t). \tag{4}$$

From the flux of electromagnetic energy density (the Poynting vector) $\mathbf{S} = \mathbf{E}\times\mathbf{H}$ we deduce the power dP radiated into the solid angle $d\Omega$ to be

$$dP = \lim_{r\to\infty} r^2 d\Omega\, \hat{\mathbf{r}}\cdot\mathbf{S}. \tag{5}$$

Using $\hat{\mathbf{r}}\cdot\mathbf{S} = \hat{\mathbf{r}}\cdot(\mathbf{E}\times\mathbf{H}) = (\hat{\mathbf{r}}\times\mathbf{E})\cdot\mathbf{H}$ show that this leads to the expression

$$\frac{\partial P}{\partial\Omega} = \lim_{r\to\infty}\frac{1}{4\pi}\left(\frac{\mu_0 c}{4\pi}\right)\left|\frac{c\mathbf{B}(\mathbf{r},t)}{\frac{\mu_0 c}{4\pi}\frac{1}{r}}\right|^2 = \lim_{r\to\infty}\frac{1}{4\pi}\left(\frac{\mu_0 c}{4\pi}\right)\left|\hat{\mathbf{r}}\times\boldsymbol{\iota}\right|^2, \tag{6}$$

where we defined the effective current (with direction), using the Greek letter iota,

$$\boldsymbol{\iota}\left(\hat{\mathbf{r}}, t - \frac{r}{c}\right) = \int d^3r' \left\{\frac{1}{c}\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}',t')\right\}_{t'=t_r}. \tag{7}$$

Verify that $c\mathbf{B}(\mathbf{r},t)/\left(\frac{\mu_0 c}{4\pi}\frac{1}{r}\right)$ has the dimensions of current. Thus, conclude that

$$\frac{\mu_0 c}{4\pi} = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} = 29.9792458\,\Omega \tag{8}$$

has the dimensions of resistance.

(a) Consider an antenna configuration consisting of parallel current carrying wires of length L, separated by distance a, described by

$$\mathbf{J}(\mathbf{r}',t') = \hat{\mathbf{z}}\, I_0 \sin\omega_0 t'\, \delta\left(x' + \frac{a}{2}\right) \delta(y')\theta(-L < 2z' < L). \\ +\hat{\mathbf{z}}\, I_0 \sin\omega_0 t'\, \delta\left(x' - \frac{a}{2}\right) \delta(y')\theta(-L < 2z' < L). \tag{9}$$

The function θ equals 1 when it's argument is a true statement, and zero otherwise. Show that

$$\boldsymbol{\iota}\left(\hat{\mathbf{r}}, t - \frac{r}{c}\right) = \hat{\mathbf{z}}\, 2I_0 \cos\left(\omega_0\left(t - \frac{r}{c} - \frac{a}{2c}\sin\theta\cos\phi\right)\right) \frac{\sin\left(\pi\frac{L}{\lambda_0}\cos\theta\right)}{\cos\theta}, \\ +\hat{\mathbf{z}}\, 2I_0 \cos\left(\omega_0\left(t - \frac{r}{c} + \frac{a}{2c}\sin\theta\cos\phi\right)\right) \frac{\sin\left(\pi\frac{L}{\lambda_0}\cos\theta\right)}{\cos\theta}, \tag{10}$$

where $\omega_0/c = 2\pi/\lambda_0$. Then, evaluate the expression for the magnetic field.

(b) Using Eq. (6) show that

$$\frac{\partial P}{\partial \Omega} = P_0 \frac{\sin^2\theta}{\pi} \frac{\sin^2\left(\pi\frac{L}{\lambda_0}\cos\theta\right)}{\cos^2\theta} \left[\cos\left(\omega_0\left(t - \frac{r}{c} - \frac{a}{2c}\sin\theta\cos\phi\right)\right) \right. \\ \left. + \cos\left(\omega_0\left(t - \frac{r}{c} + \frac{a}{2c}\sin\theta\cos\phi\right)\right)\right]^2, \tag{11}$$

where

$$P_0 = \left(\frac{\mu_0 c}{4\pi}\right) I_0^2. \tag{12}$$

Evaluate the average power $\bar{P}$ radiated into a solid angle using

$$\frac{\partial \bar{P}}{\partial \Omega} = \frac{1}{T_0}\int_0^{T_0} dt\, \frac{\partial P}{\partial \Omega}, \tag{13}$$

where $\omega_0 = 2\pi/T_0$. Show that

$$\frac{\partial \bar{P}}{\partial \Omega} = P_0 \frac{\sin^2\theta}{\pi} \frac{\sin^2\left(\pi\frac{L}{\lambda_0}\cos\theta\right)}{\cos^2\theta} 2\cos^2\left(\pi\frac{a}{\lambda_0}\sin\theta\cos\phi\right). \tag{14}$$

Hint: Use the integral

$$\frac{1}{T_0}\int_0^{T_0} dt\, \cos^2(\omega_0 t + \delta) = \frac{1}{2}. \tag{15}$$

(c) For the case $L \ll \lambda_0$, use the approximation

$$\frac{\sin\left(\pi\frac{L}{\lambda_0}\cos\theta\right)}{\cos\theta} \sim \pi\frac{L}{\lambda_0} \tag{16}$$

to obtain

$$\frac{\partial \bar{P}}{\partial \Omega} = P_0 \frac{\sin^2\theta}{\pi} \left(\pi\frac{L}{\lambda_0}\right)^2 2\cos^2\left(\pi\frac{a}{\lambda_0}\sin\theta\cos\phi\right). \tag{17}$$

(d) For the case $\lambda_0 \ll L$, if we restrict our observation region to $\theta \sim \pi/2$, the system has the characteristics of a two dimensional system. To bring this characteristic out we integrate over θ,

$$\frac{\partial \bar{P}}{\partial \phi} = P_0 \int_0^\pi \sin\theta d\theta \frac{\sin^2\theta}{\pi} \frac{\sin^2\left(\pi \frac{L}{\lambda_0}\cos\theta\right)}{\cos^2\theta} 2\cos^2\left(\pi \frac{a}{\lambda_0}\sin\theta\cos\phi\right). \tag{18}$$

Substitute

$$z = \pi \frac{L}{\lambda_0}\cos\theta \tag{19}$$

such that

$$-\sin\theta d\theta = \frac{dz}{(\pi L/\lambda_0)} \tag{20}$$

and use the approximations

$$\sin\theta = \sqrt{1 - \frac{z^2}{(\pi L/\lambda_0)^2}} \sim 1 \tag{21}$$

and

$$\pi \frac{L}{\lambda_0} \to \infty \tag{22}$$

to derive

$$\frac{\partial \bar{P}}{\partial \phi} = P_0 \left(\pi \frac{L}{\lambda_0}\right) 2\cos^2\left(\pi \frac{a}{\lambda_0}\cos\phi\right) \frac{1}{\pi} \int_{-\infty}^{\infty} dz \frac{\sin^2 z}{z^2}. \tag{23}$$

Use the integral

$$\int_0^\infty dz \frac{\sin^2 z}{z^2} = \int_0^\infty dz \frac{\sin z}{z} = \frac{\pi}{2}. \tag{24}$$

Thus, derive the expression for the average power radiated per angle $d\phi$,

$$\frac{\partial \bar{P}}{\partial \phi} = P_0 \left(\pi \frac{L}{\lambda_0}\right) 2\cos^2\left(\pi \frac{a}{\lambda_0}\cos\phi\right). \tag{25}$$

Compare this with the formula for double-slit interference pattern obtained using the Huygens-Fresnel principle for the classical wave propagation of light.