Homework No. 14 (Spring 2023)

PHYS 520B: ELECTROMAGNETIC THEORY

Department of Physics, Southern Illinois University–Carbondale

Due date: Friday, 2023 May 5, 4.30pm

1. (**20 points.**) The free Green dyadic $\boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega)$ satisfies the dyadic differential equation

$$\frac{c^2}{\omega^2}\left[\boldsymbol{\nabla}\boldsymbol{\nabla} - \mathbf{1}\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\right] \cdot \boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega) = \mathbf{1}\delta^{(3)}(\mathbf{r} - \mathbf{r}'). \tag{1}$$

   (a) Show that the divergence of the free Green dyadic is

$$\boldsymbol{\nabla} \cdot \boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega) = -\boldsymbol{\nabla}\delta^{(3)}(\mathbf{r} - \mathbf{r}'). \tag{2}$$

   (b) Substitute the divergence in the dyadic differential equation and derive

$$-\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega) = \left(\boldsymbol{\nabla}\boldsymbol{\nabla} + \frac{\omega^2}{c^2}\mathbf{1}\right)\delta^{(3)}(\mathbf{r} - \mathbf{r}'). \tag{3}$$

   (c) Construct the differential equation

$$-(\nabla^2 + k^2)G_0(\mathbf{r}, \mathbf{r}'; \omega) = \delta^{(3)}(\mathbf{r} - \mathbf{r}') \tag{4}$$

   for the Green function $G_0(\mathbf{r}, \mathbf{r}'; \omega)$, where

$$k = \frac{\omega}{c}. \tag{5}$$

   The free Green function has the (causal) solution

$$G_0(\mathbf{r} - \mathbf{r}'; \omega) = \frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}. \tag{6}$$

   Show that the free Green dyadic can be expressed in terms of the free Green function as

$$\boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega) = \left[\boldsymbol{\nabla}\boldsymbol{\nabla} + k^2\mathbf{1}\right]G_0(\mathbf{r}, \mathbf{r}'; \omega) \tag{7}$$

   (d) The free Green dyadic is a function of $\mathbf{r} - \mathbf{r}'$. Thus, we can choose $\mathbf{r}'$ to be the origin without any loss of generality. Substituting $\mathbf{r} \to \mathbf{r} - \mathbf{r}'$ at any moment of the calculation returns the dependence in $\mathbf{r}'$. Evaluate the gradient operators and show that, for $\mathbf{r}' = 0$,

$$\boldsymbol{\Gamma}_0(\mathbf{r}; \omega) = \frac{e^{ikr}}{4\pi r^3}\Big[-u(ikr)\mathbf{1} + v(ikr)\hat{\mathbf{r}}\hat{\mathbf{r}}\Big], \tag{8}$$

   where

$$u(x) = 1 - x + x^2, \tag{9a}$$
$$v(x) = 3 - 3x + x^2. \tag{9b}$$

2. (**20 points.**) The free Green dyadic $\boldsymbol{\Gamma}_0$ can be expressed in terms of the free Green function G_0 as

$$\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega) = \big[\boldsymbol{\nabla}\boldsymbol{\nabla} + k^2\mathbf{1}\big] G_0(\mathbf{r},\mathbf{r}';\omega), \tag{10}$$

where

$$G_0(\mathbf{r}-\mathbf{r}';\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{11}$$

In the far-field approximation,

$$r' \ll r, \tag{12}$$

when the observation point $\mathbf{r}$ is very far relative to the source point $\mathbf{r}'$, show that

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr'} \sim r - \hat{\mathbf{r}}\cdot\mathbf{r}'. \tag{13}$$

Thus, in the far-field asymptotic limit show that

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \to \frac{e^{ikr}}{4\pi r}\, e^{-i\mathbf{k}'\cdot\mathbf{r}'}, \tag{14}$$

where we introduced the notation

$$\mathbf{k}' = k\,\hat{\mathbf{r}}. \tag{15}$$

Further, the far-field approximation allows the replacement

$$\boldsymbol{\nabla} \to i\mathbf{k}'. \tag{16}$$

Thus, in the far-field approximation show that

$$(\boldsymbol{\nabla}\boldsymbol{\nabla} + k^2\mathbf{1}) \to (\mathbf{1} - \hat{\mathbf{r}}\hat{\mathbf{r}})k^2 = -\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{1})k^2, \tag{17}$$

which projects vectors in the plane normal to the radial direction. Thus, show that the free Green dyadic in the far-field approximation takes the form

$$\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega) = -\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{1})\frac{k^2}{4\pi}\frac{e^{ikr}}{r}\, e^{-i\mathbf{k}'\cdot\mathbf{r}'}. \tag{18}$$