Final Exam (Fall 2023)

PHYS 500A: MATHEMATICAL METHODS

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1. (**20 points.**) Find the roots of the equation

$$z^4 + 1 = 0. \tag{1}$$

Mark the points corresponding to the roots on the complex plane.

2. (**20 points.**) The generating function for the Legendre polynomials $P_l(x)$ of degree l is

$$g(t,x) = \frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{2}$$

Evaluate $P_{11}(0)$ and $P_{12}(0)$.

3. (**20 points.**) The generating function for the Legendre polynomials $P_l(x)$ of degree l is

$$g(t,x) = \frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{3}$$

(a) Starting from the generating function and differentiating with respect to t we derived the recurrence relation for Legendre polynomials in Eq. (4),

$$(l+1)\,P_{l+1}(x) = (2l+1)x\,P_l(x) - l\,P_{l-1}(x), \qquad l = 0,1,2,\ldots, \tag{4}$$

in terms of

$$P_0(x) = 1 = g(0,x). \tag{5}$$

Differentiating the recurrence relation with respect to x show that

$$(2l+1)\,P_l + (2l+1)x\,P_l' = l\,P_{l-1}' + (l+1)\,P_{l+1}', \qquad l = 0,1,2,\ldots, \tag{6}$$

where we supressed the dependence in x and prime in the superscript of $P_l'(x)$ denotes derivative with respect to the argument x.

(b) Differentiating the generating function with respect to x show that

$$\frac{\partial g}{\partial x} = \frac{t}{(1+t^2-2xt)^{\frac{3}{2}}} = \sum_{l=0}^{\infty} t^l P_l'(x). \tag{7}$$

Show that the second equality can be rewritten in the form

$$\frac{t}{\sqrt{1+t^2-2xt}} = (1+t^2-2xt) \sum_{l=0}^{\infty} t^l P_l'(x), \tag{8}$$

and implies

$$t \sum_{l=0}^{\infty} t^l P_l(x) = (1+t^2-2xt) \sum_{l=0}^{\infty} t^l P_l'(x). \tag{9}$$

Express this in the form

$$\begin{aligned} t^0 \Big[P_0'(x) \Big] \quad &+ t^1 \Big[P_1'(x) - 2x P_0'(x) - P_0(x) \Big] \\ &+ \sum_{l=2}^{\infty} t^l \Big[P_l'(x) + P_{l-2}'(x) - 2x\, P_{l-1}'(x) - P_{l-1}(x) \Big] = 0. \end{aligned} \tag{10}$$

Then, using the completeness property of Taylor expansion, that is, equating the coefficients of powers of t in the expansion, show that, for t^0 and t^1,

$$P_0'(x) = 0, \tag{11a}$$

$$P_1'(x) = P_0(x) = 1, \tag{11b}$$

and matching powers of t^l for $l \geq 2$ derive a recurrence relation for the derivative of Legendre polynomials as

$$2x\, P_{l-1}' + P_{l-1} = P_l' + P_{l-2}', \qquad l = 2, 3, \ldots. \tag{12}$$

Here, we shall find it convenient to use the above recurrence relations in the form

$$2x\, P_l' + P_l = P_{l+1}' + P_{l-1}', \qquad l = 1, 2, 3, \ldots, \tag{13}$$

which is obtained by setting $l \to l+1$.

(c) Equations (6) and (13) are linear set of equations for P_{l-1}' and P_{l+1}' in terms of P_l and P_l'. Solve them to find

$$P_{l+1}' = x P_l' + (l+1)\, P_l, \qquad l = 0, 1, 2, \ldots, \tag{14a}$$

$$P_{l-1}' = x\, P_l' - l\, P_l. \qquad l = 1, 2, 3, \ldots. \tag{14b}$$

(d) Using $l \to l-1$ in Eq. (14a) show that

$$P_l' = x\, P_{l-1}' + l\, P_{l-1}. \tag{15}$$

Then, substitute Eq. (14b) to obtain

$$(1-x^2)\, P_l' = l\, P_{l-1} - xl\, P_l. \tag{16}$$

Differentiate the above equation and substitute Eq. (14b) again to derive the differential equation for Legendre polynomials as

$$\left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x} + l(l+1)\right] P_l(x) = 0. \tag{17}$$

Substitute $x = \cos\theta$ to rewrite the differential equation in the form

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + l(l+1)\right] P_l(\cos\theta) = 0. \tag{18}$$

4. (**20 points.**) The generating function for the Legendre polynomials $P_l(x)$ of degree l is

$$g(t,x) = \frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{19}$$

(a) Using binomial expansion show that

$$\frac{1}{\sqrt{1-y}} = \sum_{m=0}^{\infty} y^m \frac{(2m)!}{\left[m!\,2^m\right]^2} \tag{20}$$

and

$$(2xt-t^2)^m = \sum_{n=0}^{\infty} \frac{m!}{n!(m-n)!}(2xt)^{m-n} t^{2n} (-1)^n. \tag{21}$$

Thus, show that

$$\frac{1}{\sqrt{1+t^2-2xt}} = \sum_{m=0}^{\infty}\sum_{n=0}^{m} t^{m+n} \frac{(2m)!}{m!n!(m-n)!2^{m+n}} x^{m-n}(-1)^n. \tag{22}$$

(b) In Figure 1 we illustrate how we change the double sum in m and n to variables l and s. This is achieved using the substitutions

$$m+n = l, \tag{23a}$$

$$m-n = 2s, \tag{23b}$$

which corresponds to

$$2m = l+2s, \qquad m = \frac{l}{2}+s, \quad \text{and} \quad n = \frac{l}{2}-s. \tag{24}$$

The counting on the variable s, for given l, follows the pattern,

$$l \ \text{even}: \quad 2s = 0,2,4,\ldots,l, \tag{25a}$$

$$l \ \text{odd}: \quad 2s = 1,3,5,\ldots,l. \tag{25b}$$

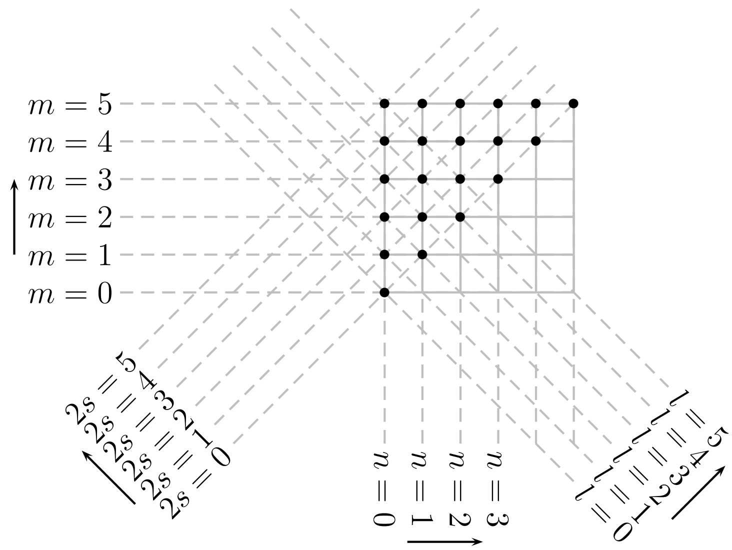

Figure 1: Double summation.

Show that in terms of l and s the double summation can be expressed as

$$\frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty}\sum_{s} t^l \frac{(l+2s)!}{\left(\frac{l}{2}+s\right)!\left(\frac{l}{2}-s\right)!(2s)!2^l} x^{2s}(-1)^{\frac{l}{2}-s}, \tag{26}$$

where the limits on the sum in s are dictated by Eqs. (25) depending on l being even or odd. Thus, read out the polynomial expression for Legendre polynomials of degree l to be

$$P_l(x) = \sum_{s} \frac{(l+2s)!}{\left(\frac{l}{2}+s\right)!\left(\frac{l}{2}-s\right)!(2s)!2^l} x^{2s}(-1)^{\frac{l}{2}-s}, \tag{27}$$

where the summation on s depends on whether l is even or odd.

(c) Show that

$$\left(\frac{d}{dx}\right)^l x^{l+2s} = \frac{(l+2s)!}{(2s)!} x^{2s}. \tag{28}$$

Thus, show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{s} \frac{l!}{\left(\frac{l}{2}+s\right)!\left(\frac{l}{2}-s\right)!} x^{l+2s}(-1)^{\frac{l}{2}-s}. \tag{29}$$

(d) For even l the summation in s runs from $s=0$ to $s=l/2$, Thus, writing $l+2s = 2[l-\left(\frac{l}{2}-s\right)]$, show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{s=0}^{\frac{l}{2}} \frac{l!}{\left(\frac{l}{2}+s\right)!\left(\frac{l}{2}-s\right)!} (x^2)^{l-\left(\frac{l}{2}-s\right)}(-1)^{\left(\frac{l}{2}-s\right)}. \tag{30}$$

Then, substituting

$$\frac{l}{2} - s = n, \tag{31}$$

show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{n=0}^{\frac{l}{2}} \frac{l!}{(l-n)!n!}(x^2)^{l-n}(-1)^n. \tag{32}$$

Note that the summation on n runs from $n = 0$ to $n = l/2$. If we were to extend this sum to $n = l$ verify that the additional terms will have powers in x less than l. Since the terms in the sum are acted upon by l derivatives with respect to x these additional terms will not contribute. Thus, show that

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{l!\,2^l}. \tag{33}$$

Similarly, for odd l the summation is s runs as

$$2s = 1, 3, 5, \ldots, l, \tag{34}$$

or

$$\frac{2s-1}{2} = 0, 1, 2, \ldots, \frac{l-1}{2}. \tag{35}$$

Thus, substituting

$$s' = \frac{2s-1}{2} = s - \frac{1}{2}, \tag{36}$$

show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{s=0}^{\frac{l-1}{2}} \frac{l!}{\left(\frac{l+1}{2}+s\right)!\left(\frac{l-1}{2}-s\right)!} x^{l+1+2s}(-1)^{\left(\frac{l-1}{2}-s\right)}. \tag{37}$$

Substituting

$$\frac{l-1}{2} - s = n \tag{38}$$

and writing

$$\frac{l+1}{2} + s = l - \left(\frac{l-1}{2} - s\right) \tag{39}$$

show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{n=0}^{\frac{l-1}{2}} \frac{l!}{(l-n)!n!}(x^2)^{l-n}(-1)^n. \tag{40}$$

Again, like in the case of even l we can extend the sum on n beyond $n = (l-1)/2$, because they do not survive under the action of l derivatives with respect to x. Thus, again, we have

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{l!\,2^l}, \tag{41}$$

which is exactly the form obtained for even l. The expression in Eq. (41) is the Rodrigues formula for generating the Legendre polynomials of degree l.

5. (**20 points.**) Consider the electric potential due to a solid sphere with uniform charge density Q. The angular integral in this evaluation involves the integral

$$\frac{1}{2}\int_{-1}^{1} dt \frac{1}{\sqrt{r^2 + r'^2 - 2rr't}}. \tag{42}$$

Evaluate the integral for $r < r'$ and $r' < r$, where r and r' are distances measured from the center of the sphere. (Hint: Substitute $r^2 + r'^2 - 2rr't = y$.)