Homework No. 01 (Fall 2023)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2023 Aug 28, 4.30pm

1. (**20 points.**) Verify the following relations:

$$\delta_{ij} = \delta_{ji}, \tag{1a}$$
$$\delta_{ii} = 3, \tag{1b}$$
$$\delta_{ik}\delta_{kj} = \delta_{ij}, \tag{1c}$$
$$\delta_{im}B_m = B_i, \tag{1d}$$
$$\varepsilon_{ijk} = -\varepsilon_{ikj} = \varepsilon_{kij}, \tag{1e}$$
$$\varepsilon_{iik} = 0, \tag{1f}$$
$$\delta_{ij}\varepsilon_{ijk} = 0. \tag{1g}$$

2. (**20 points.**) In three dimensions the Levi-Civita symbol is given in terms of the determinant of the Kronecker δ-functions,

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \tag{2a}$$
$$= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}). \tag{2b}$$

Using the above identity show that

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}, \tag{3a}$$
$$\varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn}, \tag{3b}$$
$$\varepsilon_{ijk}\varepsilon_{ijk} = 6. \tag{3c}$$

3. (**20 points.**) Use index notation or dyadic notation to show that

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A}, \tag{4a}$$
$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B}), \tag{4b}$$
$$\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} + \mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{B}) - (\boldsymbol{\nabla} \cdot \mathbf{A})\mathbf{B} - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B}. \tag{4c}$$

4. (**20 points.**) Consider the dyadic construction

$$\mathbf{T} = \mathbf{E}\,\mathbf{B} \tag{5}$$

built using the vector fields,

$$\mathbf{E} = E_x \,\hat{\mathbf{x}} + E_y \,\hat{\mathbf{y}}, \tag{6a}$$

$$\mathbf{B} = B \,\hat{\mathbf{y}}. \tag{6b}$$

Evaluate the following components of the dyadic:

$$\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} = \tag{7a}$$

$$\hat{\mathbf{y}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \hat{\mathbf{y}} \cdot \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \hat{\mathbf{y}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} = \tag{7b}$$

$$\hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} = \tag{7c}$$

Evaluate the scalars

$$\mathrm{Tr}(\mathbf{T}) = T_{ii}, \tag{8a}$$

$$\mathrm{Tr}(\mathbf{T} \cdot \mathbf{T}) = T_{ij} T_{ji}, \tag{8b}$$

$$\mathrm{Tr}(\mathbf{T} \cdot \mathbf{T} \cdot \mathbf{T}) = T_{ij} T_{jk} T_{ki}. \tag{8c}$$

Evaluate the following vector field constructions:

$$\hat{\mathbf{x}} \cdot \mathbf{T} = \qquad \hat{\mathbf{y}} \cdot \mathbf{T} = \qquad \hat{\mathbf{z}} \cdot \mathbf{T} = \tag{9a}$$

$$\mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \mathbf{T} \cdot \hat{\mathbf{z}} = \tag{9b}$$

$$\hat{\mathbf{x}} \times \mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \hat{\mathbf{y}} \times \mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \hat{\mathbf{z}} \times \mathbf{T} \cdot \hat{\mathbf{x}} = \tag{9c}$$

$$\hat{\mathbf{x}} \times \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \hat{\mathbf{y}} \times \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \hat{\mathbf{z}} \times \mathbf{T} \cdot \hat{\mathbf{y}} = \tag{9d}$$

$$\hat{\mathbf{x}} \times \mathbf{T} \cdot \hat{\mathbf{z}} = \qquad \hat{\mathbf{y}} \times \mathbf{T} \cdot \hat{\mathbf{z}} = \qquad \hat{\mathbf{z}} \times \mathbf{T} \cdot \hat{\mathbf{z}} = \tag{9e}$$

$$\hat{\mathbf{x}} \cdot \mathbf{T} \times \hat{\mathbf{x}} = \qquad \hat{\mathbf{x}} \cdot \mathbf{T} \times \hat{\mathbf{y}} = \qquad \hat{\mathbf{x}} \cdot \mathbf{T} \times \hat{\mathbf{z}} = \tag{9f}$$

$$\hat{\mathbf{y}} \cdot \mathbf{T} \times \hat{\mathbf{x}} = \qquad \hat{\mathbf{y}} \cdot \mathbf{T} \times \hat{\mathbf{y}} = \qquad \hat{\mathbf{y}} \cdot \mathbf{T} \times \hat{\mathbf{z}} = \tag{9g}$$

$$\hat{\mathbf{z}} \cdot \mathbf{T} \times \hat{\mathbf{x}} = \qquad \hat{\mathbf{z}} \cdot \mathbf{T} \times \hat{\mathbf{y}} = \qquad \hat{\mathbf{z}} \cdot \mathbf{T} \times \hat{\mathbf{z}} = \tag{9h}$$