Homework No. 03 (Fall 2023)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Friday, 2023 Sep 15, 4.30pm

0. Problems 1 and 4 are for practice. Problem 2 and 3 are for submission.

1. (**Example.**) Let $\mathbf{r}$ represent the position vector, x^i the components of the position vector in rectangular coordinates, and u^i the components of the position vector in cylindrical polar coordinates. In particular, we have

$$x^1 = x = \rho\cos\phi, \qquad u^1 = \rho = \sqrt{x^2+y^2}, \tag{1a}$$
$$x^2 = y = \rho\sin\phi, \qquad u^2 = \phi = \tan^{-1}\frac{y}{x}, \tag{1b}$$
$$x^3 = z = z, \qquad u^3 = z = z. \tag{1c}$$

Let us define the unit vectors

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2a}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2b}$$
$$\hat{\mathbf{z}} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + \hat{\mathbf{k}}, \tag{2c}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are basis vectors in rectangular coordinate system. We will also use the notation $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}^3 = \hat{\mathbf{x}}_3$, to represent these vectors.

   (a) Basis vectors:
   $$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i}. \tag{3}$$
   Show that
   $$\mathbf{e}_1 = \hat{\boldsymbol{\rho}}, \qquad \mathbf{e}_2 = \rho\hat{\boldsymbol{\phi}}, \qquad \mathbf{e}_3 = \hat{\mathbf{z}}. \tag{4}$$

   (b) Reciprocal basis vectors:
   $$\mathbf{e}^i = \boldsymbol{\nabla} u^i. \tag{5}$$
   Show that
   $$\mathbf{e}^1 = \hat{\boldsymbol{\rho}}, \qquad \mathbf{e}^2 = \frac{\hat{\boldsymbol{\phi}}}{\rho}, \qquad \mathbf{e}^3 = \hat{\mathbf{z}}. \tag{6}$$

   (c) Orthonormality: Show that
   $$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j. \tag{7}$$

(d) Metric tensor: A line element is defined as

$$d\mathbf{r} = dx^i \hat{\mathbf{x}}_i = du^i \mathbf{e}_i. \tag{8}$$

Show that

$$d\mathbf{r} \cdot d\mathbf{r} = du^i du^j g_{ij}, \tag{9}$$

where the metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \tag{10}$$

Evaluate all the components of g_{ij}.

(e) Completeness relation: Starting from

$$\boldsymbol{\nabla}\mathbf{r} = \mathbf{1} \tag{11}$$

derive the completeness relation

$$\mathbf{e}^i \mathbf{e}_i = \mathbf{1}. \tag{12}$$

Express the completeness relation in cylindrical polar coordinates in terms of $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$.

(f) Transformation matrix: The components of a vector $\mathbf{A}$ are defined using the relations

$$\mathbf{A} = \mathbf{A} \cdot \mathbf{1} = A^i \hat{\mathbf{x}}_i = \bar{A}^i \mathbf{e}_i, \tag{13a}$$

$$\mathbf{A} = \mathbf{1} \cdot \mathbf{A} = \hat{\mathbf{x}}^i A_i = \mathbf{e}^i \bar{A}_i. \tag{13b}$$

Then, derive the transformations

$$\bar{A}^j = A^i T_i^{\ j}, \qquad T_i^{\ j} = \hat{\mathbf{x}}_i \cdot \mathbf{e}^j, \tag{14a}$$

$$\bar{A}_j = S_j^{\ i} A_i, \qquad S_j^{\ i} = \mathbf{e}_j \cdot \hat{\mathbf{x}}^i, \tag{14b}$$

and show that $T_i^{\ j} S_j^{\ k} = \delta_i^k$. Find S and T for cylindrical polar coordinates.

2. (**Example.**) Let $\mathbf{r}$ represent the position vector, x^i the components of the position vector in rectangular coordinates, and u^i the components of the position vector in spherical polar coordinates. In particular, we have

$$x^1 = x = r \sin\theta \cos\phi, \qquad u^1 = r = \sqrt{x^2 + y^2 + z^2}, \tag{15a}$$

$$x^2 = y = r \sin\theta \sin\phi, \qquad u^2 = \theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}, \tag{15b}$$

$$x^3 = z = r \cos\theta, \qquad u^3 = \phi = \tan^{-1} \frac{y}{x}. \tag{15c}$$

Let us define the unit vectors

$$\hat{\mathbf{r}} = \sin\theta \cos\phi \,\hat{\mathbf{i}} + \sin\theta \sin\phi \,\hat{\mathbf{j}} + \cos\theta \,\hat{\mathbf{k}}, \tag{16a}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta \cos\phi \,\hat{\mathbf{i}} + \cos\theta \sin\phi \,\hat{\mathbf{j}} - \sin\theta \,\hat{\mathbf{k}}, \tag{16b}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi \,\hat{\mathbf{i}} + \cos\phi \,\hat{\mathbf{j}}, \tag{16c}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are basis vectors in rectangular coordinate system. We will also use the notation $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}^3 = \hat{\mathbf{x}}_3$, to represent these vectors.

(a) Basis vectors:

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i}. \tag{17}$$

Show that

$$\mathbf{e}_1 = \hat{\mathbf{r}}, \qquad \mathbf{e}_2 = r\hat{\boldsymbol{\theta}}, \qquad \mathbf{e}_3 = r\sin\theta\hat{\boldsymbol{\phi}}. \tag{18}$$

(b) Reciprocal basis vectors:

$$\mathbf{e}^i = \boldsymbol{\nabla} u^i. \tag{19}$$

Show that

$$\mathbf{e}^1 = \hat{\mathbf{r}}, \qquad \mathbf{e}^2 = \frac{\hat{\boldsymbol{\theta}}}{r}, \qquad \mathbf{e}^3 = \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}. \tag{20}$$

(c) Orthonormality: Show that

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j. \tag{21}$$

(d) Metric tensor: A line element is defined as

$$d\mathbf{r} = dx^i \hat{\mathbf{x}}_i = du^i \mathbf{e}_i. \tag{22}$$

Show that

$$d\mathbf{r} \cdot d\mathbf{r} = du^i du^j g_{ij}, \tag{23}$$

where the metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \tag{24}$$

Evaluate all the components of g_{ij}.

(e) Completeness relation: Starting from

$$\boldsymbol{\nabla}\mathbf{r} = \mathbf{1} \tag{25}$$

derive the completeness relation

$$\mathbf{e}^i \mathbf{e}_i = \mathbf{1}. \tag{26}$$

Express the completeness relation in spherical polar coordinates in terms of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$.

(f) Transformation matrix: The components of a vector $\mathbf{A}$ are defined using the relations

$$\mathbf{A} = \mathbf{A} \cdot \mathbf{1} = A^i \hat{\mathbf{x}}_i = \bar{A}^i \mathbf{e}_i, \tag{27a}$$

$$\mathbf{A} = \mathbf{1} \cdot \mathbf{A} = \hat{\mathbf{x}}^i A_i = \mathbf{e}^i \bar{A}_i. \tag{27b}$$

Then, derive the transformations

$$\bar{A}^j = A^i T_i^{\ j}, \qquad T_i^{\ j} = \hat{\mathbf{x}}_i \cdot \mathbf{e}^j, \tag{28a}$$

$$\bar{A}_j = S_j^{\ i} A_i, \qquad S_j^{\ i} = \mathbf{e}_j \cdot \hat{\mathbf{x}}^i, \tag{28b}$$

and show that $T_i^{\ j} S_j^{\ k} = \delta_i^k$. Find S and T for spherical polar coordinates.

3. (**20 points.**) Let $\mathbf{r}$ represent a position vector in three dimensional space. Let x^i be the components of the position vector in rectangular coordinates, which can be interpreted as surfaces of constant x^i. Let us coordinatize the space using the planes, labeled using β,

$$y = mx + \beta \tag{29}$$

where m is fixed, instead of planes with constant y. The other two sets of planes of constant x and constant z are the same. See Fig. 1. Let u^i be the components of the position vector in this new coordinatization of space. In particular, we have

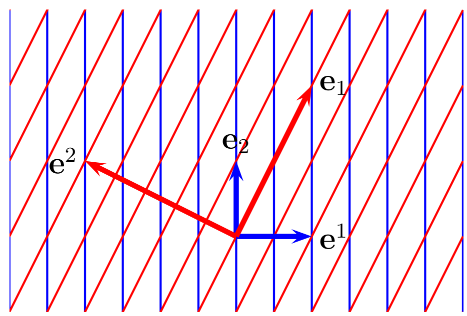

Figure 1: Basis vectors $\mathbf{e}_i$ and reciprocal basis vectors $\mathbf{e}^i$.

$$x^1 = x = \alpha, \qquad u^1 = \alpha = x, \tag{30a}$$
$$x^2 = y = mx + \beta, \qquad u^2 = \beta = y - mx, \tag{30b}$$
$$x^3 = z = \gamma, \qquad u^3 = \gamma = z. \tag{30c}$$

The basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ in rectangular coordinate system will be represented as $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}^3 = \hat{\mathbf{x}}_3$, if necessary.

(a) Basis vectors:

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i}. \tag{31}$$

Show that

$$\mathbf{e}_1 = \hat{\mathbf{i}} + m\,\hat{\mathbf{j}}, \qquad \mathbf{e}_2 = \hat{\mathbf{j}}, \qquad \mathbf{e}_3 = \hat{\mathbf{k}}. \tag{32}$$

(b) Reciprocal basis vectors:

$$\mathbf{e}^i = \boldsymbol{\nabla} u^i. \tag{33}$$

Show that

$$\mathbf{e}^1 = \hat{\mathbf{i}}, \qquad \mathbf{e}^2 = -m\,\hat{\mathbf{i}} + \hat{\mathbf{j}}, \qquad \mathbf{e}^3 = \hat{\mathbf{k}}. \tag{34}$$

Verify the relations

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{(\mathbf{e}_2 \times \mathbf{e}_3) \cdot \mathbf{e}_1}, \qquad \mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{(\mathbf{e}_3 \times \mathbf{e}_1) \cdot \mathbf{e}_2}, \qquad \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3}. \tag{35}$$

(c) Orthonormality: Show that

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j. \tag{36}$$

That is,

$$\mathbf{e}^1 \cdot \mathbf{e}_1 = 1, \qquad \mathbf{e}^1 \cdot \mathbf{e}_2 = 0, \qquad \mathbf{e}^1 \cdot \mathbf{e}_3 = 0, \tag{37a}$$
$$\mathbf{e}^2 \cdot \mathbf{e}_1 = 0, \qquad \mathbf{e}^2 \cdot \mathbf{e}_2 = 1, \qquad \mathbf{e}^2 \cdot \mathbf{e}_3 = 0, \tag{37b}$$
$$\mathbf{e}^3 \cdot \mathbf{e}_1 = 0, \qquad \mathbf{e}^3 \cdot \mathbf{e}_2 = 0, \qquad \mathbf{e}^3 \cdot \mathbf{e}_3 = 1. \tag{37c}$$

(d) Metric tensor: The metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \tag{38}$$

Evaluate all the components of g_{ij}. That is,

$$g_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1 + m^2, \qquad g_{12} = \mathbf{e}_1 \cdot \mathbf{e}_2 = m, \qquad g_{13} = \mathbf{e}_1 \cdot \mathbf{e}_3 = 0, \tag{39a}$$
$$g_{21} = \mathbf{e}_2 \cdot \mathbf{e}_1 = m, \qquad g_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1, \qquad g_{23} = \mathbf{e}_2 \cdot \mathbf{e}_3 = 0, \tag{39b}$$
$$g_{31} = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0, \qquad g_{32} = \mathbf{e}_3 \cdot \mathbf{e}_2 = 0, \qquad g_{33} = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1. \tag{39c}$$

Similarly evaluate the components of

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j. \tag{40}$$

That is,

$$g^{11} = \mathbf{e}^1 \cdot \mathbf{e}^1 = 1, \qquad g^{12} = \mathbf{e}^1 \cdot \mathbf{e}^2 = -m, \qquad g^{13} = \mathbf{e}^1 \cdot \mathbf{e}^3 = 0, \tag{41a}$$
$$g^{21} = \mathbf{e}^2 \cdot \mathbf{e}^1 = -m, \qquad g^{22} = \mathbf{e}^2 \cdot \mathbf{e}^2 = 1 + m^2, \qquad g^{23} = \mathbf{e}^2 \cdot \mathbf{e}^3 = 0, \tag{41b}$$
$$g^{31} = \mathbf{e}^3 \cdot \mathbf{e}^1 = 0, \qquad g^{32} = \mathbf{e}^3 \cdot \mathbf{e}^2 = 0, \qquad g^{33} = \mathbf{e}^3 \cdot \mathbf{e}^3 = 1. \tag{41c}$$

Verify that $g^{ij} g_{jk} = \delta^i_k$.

(e) Completeness relation: Verify the completeness relation

$$\mathbf{e}^i \mathbf{e}_i = \mathbf{1} \tag{42}$$

by evaluating

$$\mathbf{e}^1 \mathbf{e}_1 + \mathbf{e}^2 \mathbf{e}_2 + \mathbf{e}^3 \mathbf{e}_3. \tag{43}$$

(f) Given a vector

$$\mathbf{A} = a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}} \tag{44}$$

in rectangular coordinates, find the components of the vector $\mathbf{A}$ in the basis of $\mathbf{e}_i$. That is, find the components A^i in

$$\mathbf{A} = A^1\,\mathbf{e}_1 + A^2\,\mathbf{e}_2 + A^3\,\mathbf{e}_3. \tag{45}$$

4. (**20 points.**) The tangent and normal vectors for the cylindrical coordinate system are

$$\mathbf{e}_1 = \mathbf{e}_\rho = \hat{\boldsymbol{\rho}}, \qquad \mathbf{e}^1 = \mathbf{e}^\rho = \hat{\boldsymbol{\rho}}, \tag{46a}$$

$$\mathbf{e}_2 = \mathbf{e}_\phi = \rho\hat{\boldsymbol{\phi}}, \qquad \mathbf{e}^2 = \mathbf{e}^\phi = \frac{\hat{\boldsymbol{\phi}}}{\rho}, \tag{46b}$$

$$\mathbf{e}_3 = \mathbf{e}_z = \hat{\mathbf{z}}, \qquad \mathbf{e}^3 = \mathbf{e}^z = \hat{\mathbf{z}}. \tag{46c}$$

The connection is defined as

$$(\boldsymbol{\nabla}\mathbf{e}_i). \tag{47}$$

Berry connection ${\mathbf{A}_i}^k$ captures the projections of the connection to the right,

$${\mathbf{A}_i}^k = (\boldsymbol{\nabla}\mathbf{e}_i)\cdot\mathbf{e}^k. \tag{48}$$

Compute the Berry connection ${\mathbf{A}_i}^k$ for the cylindrical coordinate system to be

$${\mathbf{A}_i}^k = \begin{pmatrix} 0 & \dfrac{\hat{\boldsymbol{\phi}}}{\rho^2} & 0 \\ -\hat{\boldsymbol{\phi}} & \dfrac{\boldsymbol{\rho}}{\rho^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{49}$$

The Christoffel symbols Γ^k_{ij} captures all the projections of the connection,

$$\mathbf{e}_j\cdot(\boldsymbol{\nabla}\mathbf{e}_i)\cdot\mathbf{e}^k = \Gamma^k_{ij}. \tag{50}$$

Compute the Christoffel symbols for the cylindrical coordinate system. Show that the non-zero Christoffel symbols in cylindrical coordinates are

$$\Gamma^1_{22} = \Gamma^\rho_{\phi\phi} = -\rho, \tag{51}$$

$$\Gamma^2_{12} = \Gamma^\phi_{\rho\phi} = \frac{1}{\rho}, \tag{52}$$

$$\Gamma^2_{21} = \Gamma^\phi_{\phi\rho} = \frac{1}{\rho}. \tag{53}$$