Homework No. 05 (Fall 2023)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Friday, 2023 Sep 29, 4.30pm

1. (**20 points.**) Let $\mathbf{e}^i$, for $i = 1, 2$, be an eigenbasis set, such that

$$\mathbf{e}^1\mathbf{e}_1 + \mathbf{e}^2\mathbf{e}_2 = \mathbf{1}. \tag{1}$$

We have

$$\mathbf{e}_i = (\mathbf{e}^i)^\dagger. \tag{2}$$

Let $\mathbf{g}^i$, for $i = 1, 2$, be another eigenbasis set, such that

$$\mathbf{g}^1\mathbf{g}_1 + \mathbf{g}^2\mathbf{g}_2 = \mathbf{1}. \tag{3}$$

Using the above two eigenbases we can construct the operator

$$\mathbf{U} = \mathbf{g}^1\mathbf{e}_1 + \mathbf{g}^2\mathbf{e}_2. \tag{4}$$

Show that

$$\mathbf{U}\cdot\mathbf{e}^1 = \mathbf{g}^1, \tag{5a}$$

$$\mathbf{U}\cdot\mathbf{e}^2 = \mathbf{g}^2. \tag{5b}$$

Thus, the operator $\mathbf{U}$ is the transformation matrix taking us from the eigenbasis $\mathbf{e}^i$ to $\mathbf{g}^i$. What is the associated inverse transformation? To this end, using

$$(\mathbf{g}^i\cdot\mathbf{e}_j)^\dagger = \mathbf{e}_j^\dagger\cdot\mathbf{g}^{i\dagger} = \mathbf{e}^j\cdot\mathbf{g}_i \tag{6}$$

show that

$$\mathbf{U}^\dagger = \mathbf{e}^1\mathbf{g}_1 + \mathbf{e}^2\mathbf{g}_2, \tag{7}$$

and verify

$$\mathbf{U}^\dagger\cdot\mathbf{g}^1 = \mathbf{e}^1, \tag{8a}$$

$$\mathbf{U}^\dagger\cdot\mathbf{g}^2 = \mathbf{e}^2. \tag{8b}$$

Thus, $\mathbf{U}^\dagger$ is the inverse transformation matrix taking us from the eigenbasis $\mathbf{g}^i$ back to $\mathbf{e}^i$. Further, verify that

$$\mathbf{U}^\dagger\cdot\mathbf{U} = \mathbf{1}, \tag{9a}$$

$$\mathbf{U}\cdot\mathbf{U}^\dagger = \mathbf{1}, \tag{9b}$$

which states that $\mathbf{U}$ is an unitary operator. Repeat this for i running from 1 to 3. Does it go through? Repeat this for i running from 1 to n. Does it go through?

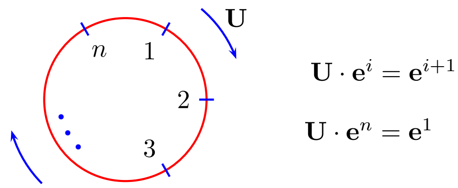

Figure 1: A periodic unitary operator.

2. (**20 points.**) Given two eigenbases $\mathbf{e}^i$ and $\mathbf{g}^i$ we can construct an unitary operator

$$\mathbf{U} = \mathbf{g}^i \mathbf{e}_i. \tag{10}$$

A Schwinger periodic unitary operator is constructed by choosing the second eigenbasis $\mathbf{g}^i$ to be the same as the original eigenbasis set $\mathbf{e}^i$ but in a different order. That is,

$$\mathbf{g}^i = \begin{cases} \mathbf{e}^{i+1}, & \text{if} \quad i \neq n, \\ \mathbf{e}^1, & \text{if} \quad i = n, \end{cases} \tag{11}$$

such that the unitary operator is

$$\mathbf{U} = \mathbf{e}^{i+1} \mathbf{e}_i. \tag{12}$$

(a) Show that

$$\mathbf{U} \cdot \mathbf{e}^i = \begin{cases} \mathbf{e}^{i+1}, & \text{if} \quad i \neq n, \\ \mathbf{e}^1, & \text{if} \quad i = n. \end{cases} \tag{13}$$

Verify that the transformation is cyclic, that is,

$$\mathbf{U} \cdot \mathbf{e}^1 = \mathbf{e}^2, \tag{14a}$$
$$\mathbf{U}^2 \cdot \mathbf{e}^1 = \mathbf{e}^3, \tag{14b}$$
$$\vdots$$
$$\mathbf{U}^{n-1} \cdot \mathbf{e}^1 = \mathbf{e}^n, \tag{14c}$$
$$\mathbf{U}^n \cdot \mathbf{e}^1 = \mathbf{e}^1. \tag{14d}$$

The cyclic nature of the transformation of the eigenbasis is illustrated in Figure 1 and generated by the periodic unitary operator $\mathbf{U}$.

(b) Show that the periodic unitary operator satisfies

$$\mathbf{U}^n = \mathbf{1}, \tag{15}$$

where n is called the period of $\mathbf{U}$. Consider the eigenvalue equation

$$\mathbf{U} \cdot \mathbf{u}^k = u'_k \mathbf{u}^k, \qquad \text{no sum on } k. \tag{16}$$

The eigenvalues u'_k of the operator $\mathbf{U}$ satisfy

$$(u'_k)^n = 1, \tag{17}$$

where u'_k is the k-th eigenvalue, $k = 1, 2, \ldots, n$. An immediate observation, then, is that the eigenvalues are the n-th roots of unity,

$$u'_k = e^{i\frac{2\pi}{n}k}, \qquad k = 1, 2, \ldots, n. \tag{18}$$

To determine the eigenvectors $\mathbf{u}^k$ in the eigenbasis of $\mathbf{e}^i$ start by writing

$$\mathbf{u}^k = \mathbf{1} \cdot \mathbf{u}^k \tag{19a}$$

$$= \sum_{l=1}^{n} \mathbf{e}^l \mathbf{e}_l \cdot \mathbf{u}^k \tag{19b}$$

$$= \sum_{l=1}^{n} \mathbf{U}^l \cdot \mathbf{e}^n \, (\mathbf{e}_l \cdot \mathbf{u}^k). \tag{19c}$$

where we used

$$\mathbf{e}^l = \mathbf{U}^l \cdot \mathbf{e}^n. \tag{20}$$

Operating from the left leads to

$$\mathbf{u}_{\bar{k}} \cdot \mathbf{u}^k = \sum_{l=1}^{n} \mathbf{u}_{\bar{k}} \cdot \mathbf{U}^l \cdot \mathbf{e}^n \, (\mathbf{e}_l \cdot \mathbf{u}^k). \tag{21a}$$

Starting from the eigenvalue equation for $\mathbf{U}$, operate $\mathbf{U}^\dagger$ on both sides, deduce

$$\mathbf{U}^\dagger \cdot \mathbf{u}^k = (u'_k)^* \mathbf{u}^k, \qquad \text{no sum on } k, \tag{22}$$

which leads to the reciprocal statement

$$\mathbf{u}_k \cdot \mathbf{U} = u'_k \mathbf{u}_k, \qquad \text{no sum on } k. \tag{23}$$

Thus, we have

$$\mathbf{u}_k \cdot \mathbf{U}^l = e^{i\frac{2\pi}{n}kl} \mathbf{u}_k. \tag{24}$$

In conjunction with the orthogonality relations of the eigenvectors,

$$\mathbf{u}_{\bar{k}} \cdot \mathbf{u}^k = \delta_{\bar{k}}{}^k, \tag{25}$$

show that

$$\frac{\delta_{\bar{k}}{}^k}{(\mathbf{u}_{\bar{k}} \cdot \mathbf{e}^n)} = \sum_{l=1}^{n} e^{i\frac{2\pi}{n}\bar{k}l} (\mathbf{e}_l \cdot \mathbf{u}^k). \tag{26}$$

Recognize the discrete Fourier transform in the above expression. Then, invert the equations to derive

$$(\mathbf{e}_l \cdot \mathbf{u}^k) = \frac{1}{n} \frac{e^{-i\frac{2\pi}{n}kl}}{(\mathbf{u}_k \cdot \mathbf{e}^n)}. \tag{27}$$

For $l = n$, we have

$$(\mathbf{e}_l \cdot \mathbf{u}^k)(\mathbf{u}_k \cdot \mathbf{e}^n) = \frac{1}{n}, \tag{28}$$

so that

$$(\mathbf{u}_k \cdot \mathbf{e}^n) = \frac{1}{\sqrt{n}} \tag{29}$$

up to a phase, and

$$(\mathbf{e}_l \cdot \mathbf{u}^k) = \frac{1}{\sqrt{n}} e^{-i\frac{2\pi}{n}kl}. \tag{30}$$

Thus, show that the eigenvectors $\mathbf{u}^k$ in the eigenbasis of $\mathbf{e}^i$ are given by

$$\mathbf{u}^k = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} e^{-i\frac{2\pi}{n}kl} \mathbf{e}^l. \tag{31}$$

(c) Repeat the above exercise for another periodic operator constructed out of $\mathbf{u}^i$ in the form

$$\mathbf{V} = \mathbf{u}^i \mathbf{u}_{i+1}. \tag{32}$$

Show that

$$\mathbf{u}_i \cdot \mathbf{V} = \begin{cases} \mathbf{u}_{i+1}, & \text{if} \quad i \neq n, \\ \mathbf{u}_1, & \text{if} \quad i = n. \end{cases} \tag{33}$$

Repeat the steps that we carried out for $\mathbf{U}$ now for $\mathbf{V}$. Define the eigenvalue equation to act to the left, $\mathbf{v}_k \cdot \mathbf{V} = v'_k \mathbf{v}_k$. In particular, show that

$$\mathbf{V}^n = \mathbf{1}, \tag{34}$$

eigenvalues v'_k are given by

$$v'_k = e^{i\frac{2\pi}{n}k}, \qquad k = 1, 2, \ldots, n, \tag{35}$$

and the eigenvectors are

$$\mathbf{v}_k = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} e^{-i\frac{2\pi}{n}kl} \mathbf{u}_k. \tag{36}$$

(d) Show that

$$\mathbf{v}^i = \mathbf{e}^i. \tag{37}$$

Thus, recognize that the eigenvectors of the periodic operators $\mathbf{U}$ and $\mathbf{V}$ are related to each other by discrete Fourier transformation,

$$\mathbf{u}^k = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} e^{-i\frac{2\pi}{n}kl} \mathbf{v}^l, \tag{38a}$$

$$\mathbf{v}^l = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} e^{i\frac{2\pi}{n}kl} \mathbf{u}^k. \tag{38b}$$

(e) Verify that

$$\mathbf{U} \cdot \mathbf{V} \cdot \mathbf{v}^k = \mathbf{v}^{k+1} e^{i\frac{2\pi}{n}k} \tag{39}$$

and

$$\mathbf{V} \cdot \mathbf{U} \cdot \mathbf{v}^k = \mathbf{v}^{k+1} e^{i\frac{2\pi}{n}(k+1)}. \tag{40}$$

Thus, derive

$$\mathbf{U} \cdot \mathbf{V} = e^{i\frac{2\pi}{n}} \mathbf{V} \cdot \mathbf{U}. \tag{41}$$

(f) For $n = 2$ we have

$$\mathbf{U}^2 = \mathbf{1}, \qquad \mathbf{V}^2 = \mathbf{1}, \quad \text{and} \quad \mathbf{U} \cdot \mathbf{V} = -\mathbf{V} \cdot \mathbf{U}. \tag{42}$$

These constitute four independent operators, $\mathbf{1}$, $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{U} \cdot \mathbf{V}$, which have the same algebra as that of 1, σ_x, σ_y, and σ_z.