Homework No. 06 (Fall 2023)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Friday, 2023 Oct 6, 4.30pm

1. (**20 points.**) For a given complex number z, say

$$z = \sqrt{2}\, e^{i\frac{\pi}{3}}, \tag{1}$$

evaluate

$$z^2, z^3, z^4, z^5, z^6, z^7, z^8, z^9, z^{10}. \tag{2}$$

Mark all of them on the complex plane. Decipher the pattern.

2. (**20 points.**) Evaluate

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{23}. \tag{3}$$

Mark the resulting number on the complex plane.

3. (**20 points.**) Prove the identity

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}. \tag{4}$$

Use the identity

$$(2+i)(3+i) = 5 + i5. \tag{5}$$

Similarly, find y/x in the relation

$$\tan^{-1}\left(\frac{3}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{y}{x}\right). \tag{6}$$

4. (**20 points.**) Verify that

$$\sqrt{-2}\sqrt{-3} = -\sqrt{6}. \tag{7}$$

However, it is often tempting to conclude

$$\sqrt{-2}\sqrt{-3} = \sqrt{(-2)(-3)} = \sqrt{6}. \tag{8}$$

The ambiguity in the interpretation of $\sqrt{-2}$ and $\sqrt{-3}$ is (partly) removed by writing,

$$\sqrt{-2} = \left(2e^{i\pi}\right)^{\frac{1}{2}} = \sqrt{2}e^{i\frac{\pi}{2}}, \tag{9a}$$

$$\sqrt{-3} = \left(3e^{i\pi}\right)^{\frac{1}{2}} = \sqrt{3}e^{i\frac{\pi}{2}}. \tag{9b}$$

This only partly removes the ambiguity because $\sqrt{-2}$ and $\sqrt{-3}$ have two independent roots each and Eqs. (9) only identifies one of the roots, the principal root, for each. Using Eqs. (9) verify the correctness of the statement in Eq. (7) again. The above ambiguity in the interpretation and the related confusions plagued the development of ideas related to complex numbers until the geometric visualization of a complex number using Argand diagram (magnitude and direction in polar representation) was discovered by Wessel in 1797 and popularized by Argand in 1806. Without this geometric interpretation even Euler fell into the trap of concluding $\sqrt{-2}\sqrt{-3} = \sqrt{6}$. So, is the statement in Eq. (8) erroneous? No. To this end, let us remove the ambiguity completely by recognizing the multiple roots,

$$\sqrt{-2} = \left(2e^{i\pi}\right)^{\frac{1}{2}} = \sqrt{2}e^{i\frac{\pi}{2}}(1,\omega), \quad \omega = e^{i\pi}, \tag{10a}$$

$$\sqrt{-3} = \left(3e^{i\pi}\right)^{\frac{1}{2}} = \sqrt{3}e^{i\frac{\pi}{2}}(1,\omega), \quad \omega = e^{i\pi}, \tag{10b}$$

where the comma-separated quantities contribute to the multiple roots. Muliplication of the two roots of $\sqrt{-2}$ and two roots of $\sqrt{-3}$ leads to four results,

$$(1,\omega) \times (1,\omega) \to (1,\omega,\omega,\omega^2). \tag{11}$$

Using $\omega^2 = 1$ we have only two out of the four results to be independent. Thus, we have

$$\sqrt{-2}\sqrt{-3} = \sqrt{2}\sqrt{3}(1,\omega), \quad \omega = e^{i\pi}. \tag{12}$$

In summary, both the statements in Eqs. (7) and (8) are correct.