Homework No. 10 (Fall 2023)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2023 Nov 13, 4.30pm

1. (**20 points.**) A forced harmonic oscillator, in the absence of damping, constituting of a body of mass m and a spring of spring constant k, is described by

$$ma + kx = F(t), \tag{1}$$

where x is position, $v = dx/dt$ is velocity, $a = dv/dt$ is acceleration, and $F(t)$ is a driving force. Thus, we have the differential equation

$$-\left[\frac{d^2}{dt^2} + \omega_0^2\right] x(t) = A(t), \tag{2}$$

where

$$\omega_0^2 = \frac{k}{m}, \qquad A(t) = -\frac{F(t)}{m}. \tag{3}$$

Let us consider the case with initial conditions

$$x(0) = 0, \tag{4a}$$
$$\dot{x}(0) = 0. \tag{4b}$$

Verify by substitution that

$$x(t) = -\frac{1}{\omega_0}\int_0^t dt' \sin\omega_0(t-t')A(t') \tag{5}$$

is the solution.

2. (**20 points.**) Consider the differential equation

$$-\left[\frac{d^2}{dt^2} + \omega_0^2\right] x(t) = -\omega_f^2 x_f \sin\omega_f t, \tag{6}$$

with initial conditions

$$x(0) = 0, \tag{7a}$$
$$\dot{x}(0) = 0. \tag{7b}$$

Verify by substitution that

$$x(t) = -x_f \frac{\omega_f^2}{(\omega_0^2 - \omega_f^2)}\frac{1}{\omega_0}\Big[\omega_f \sin\omega_0 t - \omega_0 \sin\omega_f t\Big] \tag{8}$$

is the solution. Show that the first term in the solution, called the transient solution, is solution to the homogeneous part of the differential equation. Show that the second term in the solution, called the steady-state solution, is a particular solution to the inhomogeneous differential equation.

3. (**20 points.**) Verify that

$$\frac{d}{dz}|z| = \theta(z) - \theta(-z), \tag{9}$$

where $\theta(z) = 1$, if $z > 0$, and 0, if $z < 0$. Further, verify that

$$\frac{d^2}{dz^2}|z| = 2\,\delta(z). \tag{10}$$

Also, argue that, for a well defined function $f(z)$, the replacement

$$f(z)\delta(z) = f(0)\delta(z) \tag{11}$$

is justified. Using Eq. (9), Eq. (10), and Eq. (11), verify (by substituting the solution into the differential equation) that

$$g(z) = \frac{1}{2k}\, e^{-k|z|} \tag{12}$$

is a particular solution of the differential equation

$$\left(-\frac{d^2}{dz^2} + k^2\right) g(z) = \delta(z). \tag{13}$$