Homework No. 11 (Fall 2023)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2023 Nov 27, 4.30pm

1. (**20 points.**) Verify the identity

$$\phi\boldsymbol{\nabla}\cdot(\lambda\boldsymbol{\nabla}\psi)-\psi\boldsymbol{\nabla}\cdot(\lambda\boldsymbol{\nabla}\phi)=\boldsymbol{\nabla}\cdot[\lambda(\phi\boldsymbol{\nabla}\psi-\psi\boldsymbol{\nabla}\phi)], \tag{1}$$

which is a slight generalization of what is known as Green's second identity. Here ϕ, ψ, and λ, are position dependent functions.

2. (**20 points.**) The expression for the electric potential due to a point charge placed in front of a perfectly conducting semi-infinite slab, described by

$$\frac{\varepsilon(z)}{\varepsilon_0}=\begin{cases}\infty, & z<0,\\ 1, & 0<z,\end{cases} \tag{2}$$

is given in terms of the reduced Green function that satisfies the differential equation $(0<\{z,z'\})$

$$-\left[\frac{\partial^2}{\partial z^2}-k^2\right]\varepsilon_0 g(z,z')=\delta(z-z') \tag{3}$$

with boundary conditions requiring the reduced Green's function to vanish at $z=0$ and at $z\to\infty$.

(a) Construct the reduced Green function in the form

$$\varepsilon_0 g(z,z')=\begin{cases}Ae^{kz}+Be^{-kz}, & 0<z<z',\\ Ce^{kz}+De^{-kz}, & 0<z'<z,\end{cases} \tag{4}$$

and solve for the four coefficients, A,B,C,D, using the conditions

$$\varepsilon_0 g(0,z')=0, \tag{5a}$$

$$\varepsilon_0 g(\infty,z')=0, \tag{5b}$$

$$\varepsilon_0 g(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta}=0, \tag{5c}$$

$$\partial_z\varepsilon_0 g(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta}=-1. \tag{5d}$$

(b) Express the solution in the form

$$\varepsilon_0 g(z,z')=\frac{1}{2k}e^{-k|z-z'|}-\frac{1}{2k}e^{-k|z|}e^{-k|z'|}. \tag{6}$$

3. (**20 points.**) The expression for the electric potential due to a point charge placed in between two parallel grounded perfectly conducting semi-infinite slabs, described by

$$\frac{\varepsilon(z)}{\varepsilon_0} = \begin{cases} \infty, & z < 0, \\ 1, & 0 < z < a, \\ \infty, & a < z, \end{cases} \tag{7}$$

is given in terms of the reduced Green function that satisfies the differential equation ($0 < \{z, z'\} < a$)

$$\left[-\frac{\partial^2}{\partial z^2} + k^2\right] \varepsilon_0 g(z, z') = \delta(z - z') \tag{8}$$

with boundary conditions requiring the reduced Green's function to vanish at $z = 0$ and $z = a$.

(a) Construct the reduced Green's function in the form

$$\varepsilon_0 g(z, z') = \begin{cases} A \sinh kz + B \cosh kz, & 0 < z < z' < a, \\ C \sinh kz + D \cosh kz, & 0 < z' < z < a, \end{cases} \tag{9}$$

and solve for the four coefficients, A, B, C, D, using the conditions

$$\varepsilon_0 g(0, z') = 0, \tag{10a}$$

$$\varepsilon_0 g(a, z') = 0, \tag{10b}$$

$$\varepsilon_0 g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0, \tag{10c}$$

$$\partial_z \varepsilon_0 g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1. \tag{10d}$$

(b) After using conditions in Eqs. (10a) and (10b) show that the reduced Green's function can be expressed in the form

$$\varepsilon_0 g(z, z') = \begin{cases} A \sinh kz, & 0 < z < z' < a, \\ C' \sinh k(a - z), & 0 < z' < z < a, \end{cases} \tag{11}$$

where $C' = -C/\cosh ka$. Then, use Eqs. (10c) and (10d) to show that

$$\varepsilon_0 g(z, z') = \begin{cases} \dfrac{\sinh kz \sinh k(a - z')}{k \sinh ka}, & 0 < z < z' < a, \\[2ex] \dfrac{\sinh kz' \sinh k(a - z)}{k \sinh ka}, & 0 < z' < z < a. \end{cases} \tag{12}$$

(c) Take the limit $ka \to \infty$ in your solution above, (which corresponds to moving the slab at $z = a$ to infinity,) to obtain the reduced Green's function for a single perfectly conducting slab,

$$\lim_{ka \to \infty} \varepsilon_0 g(z, z') = \frac{1}{2k} e^{-k|z-z'|} - \frac{1}{2k} e^{-k|z|} e^{-k|z'|}. \tag{13}$$

This should serve as a check for your solution to the reduced Green's function. Hint: The hyperbolic functions here are defined as

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \text{and} \qquad \cosh x = \frac{1}{2}(e^x + e^{-x}). \tag{14}$$