Homework No. 12 (Fall 2023)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale
Due date: Monday, 2023 Dec 4, 4.30pm

1. (**20 points.**) Integral representations for the modified Bessel functions, $I_m(t)$ and $K_m(t)$, for integer m and $0 \le t < \infty$ are

$$K_m(t) = \int_0^\infty d\theta \, \cosh m\theta \, e^{-t\cosh\theta}, \tag{1a}$$

$$I_m(t) = \int_0^\pi \frac{d\phi}{\pi} \cos m\phi \, e^{t\cos\phi}. \tag{1b}$$

   (a) Using Mathematica (or your favourite graphing tool) plot $K_0(t), K_1(t), K_2(t)$ and $I_0(t), I_1(t), I_2(t)$ on the same plot. (Please do not submit hand sketched plots.)

   (b) Refer Chapter 10 of Digital Library of Mathematical Functions,

   https://dlmf.nist.gov/10

   for a comprehensive resource.

2. (**20 points.**) Show that the integral representations for the modified Bessel functions, $I_m(t)$ and $K_m(t)$, for integer m and $0 \le t < \infty$,

$$K_m(t) = \int_0^\infty d\theta \, \cosh m\theta \, e^{-t\cosh\theta}, \tag{2a}$$

$$I_m(t) = \int_0^\pi \frac{d\phi}{\pi} \cos m\phi \, e^{t\cos\phi}. \tag{2b}$$

   satisfies the differential equation for modified Bessel functions,

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + \frac{m^2}{t^2} + 1\right] \left\{ \begin{array}{c} I_m(t) \\ K_m(t) \end{array} \right\} = 0. \tag{3}$$

   Hint: Integrate by parts, after identifying

$$\left(t\cosh\theta - t^2\sinh^2\theta\right) e^{-t\cosh\theta} = -\frac{d^2}{d\theta^2} e^{-t\cosh\theta}, \tag{4a}$$

$$\left(t\cos\phi - t^2\sin^2\phi\right) e^{t\cos\phi} = -\frac{d^2}{d\phi^2} e^{t\cos\phi}. \tag{4b}$$

3. (**20 points.**) The modified Bessel functions, $I_m(t)$ and $K_m(t)$, satisfy the differential equation

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + \frac{m^2}{t^2} + 1\right]\left\{\begin{array}{c} I_m(t) \\ K_m(t) \end{array}\right\} = 0. \tag{5}$$

Derive the identity, for the Wronskian, (upto a constant C)

$$I_m(t)K_m'(t) - K_m(t)I_m'(t) = -\frac{C}{t}, \tag{6}$$

where

$$I_m'(t) \equiv \frac{d}{dt}I_m(t) \qquad \text{and} \qquad K_m'(t) \equiv \frac{d}{dt}K_m(t). \tag{7}$$

Further, determine the value of the constant C on the right hand side of Eq. (6) using the asymptotic forms for the modified Bessel functions:

$$I_m(t) \xrightarrow{t\gg 1} \frac{1}{\sqrt{2\pi}}\frac{e^t}{\sqrt{t}}, \tag{8}$$

$$K_m(t) \xrightarrow{t\gg 1} \sqrt{\frac{\pi}{2}}\frac{e^{-t}}{\sqrt{t}}. \tag{9}$$

4. (**20 points.**) The cylindrical free Green's function satisfies

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{m^2}{\rho^2} + k_z^2\right]g_m(\rho,\rho';k_z) = \frac{\delta(\rho-\rho')}{\rho}. \tag{10}$$

Integrate Eq. (10) around $\rho = \rho'$ to derive the continuity conditions:

$$g_m(\rho,\rho';k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = 0, \tag{11a}$$

$$\rho\frac{\partial}{\partial\rho}g_m(\rho,\rho';k_z)\Bigg|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = -1. \tag{11b}$$

Let us further require that

$$g_m(0,\rho';k_z) \text{ is finite}, \tag{12a}$$

$$g_m(\infty,\rho';k_z) = 0. \tag{12b}$$

Recall the Wronskian

$$I_m(t)K_m'(t) - I_m'(t)K_m(t) = -\frac{1}{t}. \tag{13}$$

Construct the solution to have the form

$$g_m(\rho,\rho') = \begin{cases} A\,I_m(k_z\rho) + B\,K_m(k_z\rho), & 0 \le \rho < \rho', \\ C\,I_m(k_z\rho) + D\,K_m(k_z\rho), & \rho' < \rho < \infty. \end{cases} \tag{14}$$

Derive the solution

$$g_m(\rho,\rho') = I_m(k_z\rho_<)K_m(k_z\rho_>), \tag{15}$$

where $\rho_< = \text{Minimum}(\rho,\rho')$ and $\rho_> = \text{Maximum}(\rho,\rho')$.

5. (**20 points.**) The cylindrical Green's function satisfies

$$\Big[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{m^2}{\rho^2}+k_z^2\Big]g_m(\rho,\rho';k_z)=\frac{\delta(\rho-\rho')}{\rho}. \tag{16}$$

Inside a perfectly conducting cylinder we require

$$g_m(0,\rho';k_z) \text{ is finite,} \tag{17a}$$

$$g_m(a,\rho';k_z)=0. \tag{17b}$$

This shields all of (electrostatics related) physics between the plates from outside. Thus, we have

$$0\le\rho\le a, \tag{18a}$$

$$0\le\rho'\le a, \tag{18b}$$

Integrate Eq. (16) around $\rho=\rho'$ to derive the continuity conditions:

$$g_m(\rho,\rho';k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta}=0, \tag{19a}$$

$$\rho\frac{\partial}{\partial\rho}g_m(\rho,\rho';k_z)\Bigg|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta}=-1. \tag{19b}$$

Recall the Wronskian

$$I_m(t)K_m'(t)-I_m'(t)K_m(t)=-\frac{1}{t}. \tag{20}$$

Construct the solution to have the form

$$g_m(\rho,\rho')=\begin{cases}A\,I_m(k_z\rho)+B\,K_m(k_z\rho), & 0\le\rho<\rho',\\ C\,I_m(k_z\rho)+D\,K_m(k_z\rho), & \rho'<\rho<a.\end{cases} \tag{21}$$

Derive the solution

$$g_m(\rho,\rho')=I_m(k_z\rho_<)K_m(k_z\rho_>)-\frac{K_m(k_za)}{I_m(k_za)}I_m(k_z\rho)I_m(k_z\rho'). \tag{22}$$