Homework No. 13 (Fall 2023)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale
Due date: Friday, 2023 Dec 8, 4.30pm

1. (**20 points.**) The Legendre polynomials are defined, or generated, by expanding the electric (or gravitational) potential of a point charge (or mass),

$$\frac{\alpha}{|\mathbf{r}-\mathbf{r}'|} = \frac{\alpha}{r_>}\frac{1}{\sqrt{1+\left(\frac{r_<}{r_>}\right)^2 - 2\left(\frac{r_<}{r_>}\right)\cos\gamma}} = \frac{\alpha}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\cos\gamma), \tag{1}$$

where

$$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}' = \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'), \tag{2}$$

and

$$r_< = \text{Minimum}(r, r'), \tag{3a}$$
$$r_> = \text{Maximum}(r, r'). \tag{3b}$$

Thus, in terms of variables

$$t = \frac{r_<}{r_>}, \qquad 0 \le t < \infty, \tag{4}$$

and

$$x = \cos\gamma, \qquad -1 \le x < 1, \tag{5}$$

we define the generating function for the Legendre polynomials as

$$g(t,x) = \frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{6}$$

The recurrence relation for Legendre polynomials can be derived by differentiating the generating function with respect to t to obtain

$$\frac{\partial g}{\partial t} = \frac{(x-t)}{(1+t^2-2xt)^{\frac{3}{2}}} = \sum_{l=1}^{\infty} l\, t^{l-1} P_l(x). \tag{7}$$

Inquire why the sum on the right hand side now starts from $l=1$. The second equality can be rewritten in the form

$$\frac{(x-t)}{\sqrt{1+t^2-2xt}} = (1+t^2-2xt)\sum_{l=1}^{\infty} l\, t^{l-1} P_l(x), \tag{8}$$

and implies

$$(x-t)\sum_{l=0}^{\infty} t^l P_l(x) = (1+t^2-2xt)\sum_{l=1}^{\infty} l\, t^{l-1} P_l(x). \tag{9}$$

Express this in the form

$$t^0\Big[xP_0(x) - P_1(x)\Big] \quad + t^1\Big[3xP_1(x) - P_0(x) - 2P_2(x)\Big]$$
$$+ \sum_{l=2}^{\infty} t^l \Big[(2l+1)x\, P_l(x) - l\, P_{l-1}(x) - (l+1)\, P_{l+1}(x)\Big] = 0. \tag{10}$$

Thus, using the completeness property of Taylor expansion, that is, expansion in powers of t, we have,

$$P_1(x) = xP_0(x), \tag{11}$$
$$2P_2(x) = 3xP_1(x) - P_0(x), \tag{12}$$

and

$$(l+1)\, P_{l+1}(x) = (2l+1)x\, P_l(x) - l\, P_{l-1}(x), \qquad l = 1, 2, 3, \ldots. \tag{13}$$

This generates Legendre polynomials of all orders starting from

$$P_0(x) = 1. \tag{14}$$

2. (**20 points.**) Using Mathematica (or another graphing tool) plot the Legendre polynomials $P_l(x)$ for $l = 0, 1, 2, 3, 4$ on the same plot. Note that $-1 \leq x \leq 1$. Based on the pattern you see what can you conclude about the number of roots for $P_l(x)$. In Mathematica these plots are generated using the following commands:
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Plot[{LegendreP[0,x], LegendreP[1,x], LegendreP[2,x], LegendreP[3,x],
LegendreP[4,x] },{x,-1,1}]
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Compare your plots with those in Wikipedia article on 'Legendre Polynomials'. While there read the Wikipedia article on Adrien-Marie Legendre and the associated 'Portrait Debacle'.

3. (**20 points.**) Legendre polynomials are conveniently generated using the relation

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}, \tag{15}$$

where $-1 \leq x \leq 1$. Evaluate Legendre polynomials of degree $l = 0, 1, 2, 3, 4$ in this manner.

4. (**20 points.**) Legendre polynomials satisfy the differential equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + l(l+1)\right] P_l(\cos\theta) = 0. \tag{16}$$

Verify this explicitly for $l = 0, 1, 2, 3, 4$.

5. (**20 points.**) Legendre polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}. \tag{17}$$

Verify this explicitly for $l = 0, 1, 2$ and $l' = 0, 1, 2$. The orthogonality relation is also expressed as

$$\int_{0}^{\pi} \sin\theta d\theta\, P_l(\cos\theta) P_{l'}(\cos\theta) = \frac{2}{2l+1}\delta_{ll'}. \tag{18}$$