Final Exam (2024 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Date: 2024 May 9

1. (**20 points.**) Evaluate the functional derivative

$$\frac{\delta F[u]}{\delta u(x)} \tag{1}$$

of the following functional,

$$F[u] = \int_a^b dx \sqrt{1 + \left(\frac{du}{dx}\right)^2}, \tag{2}$$

assuming no variation at the end points.

2. (**20 points.**) A mass m slides down a frictionless ramp that is inclined at an angle θ with respect to the horizontal. Assume uniform acceleration due to gravity g in the vertical downward direction. In terms of a suitable dynamical variable write a Lagrangian that describes the motion of the mass.

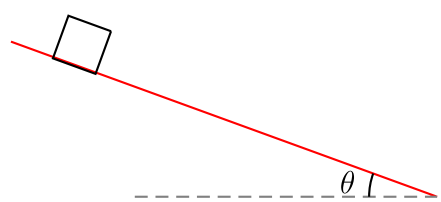

Figure 1: Problem 2.

3. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration α is described by equation of a hyperbola

$$z^2 - c^2t^2 = z_0^2, \qquad z_0 = \frac{c^2}{\alpha}. \tag{3}$$

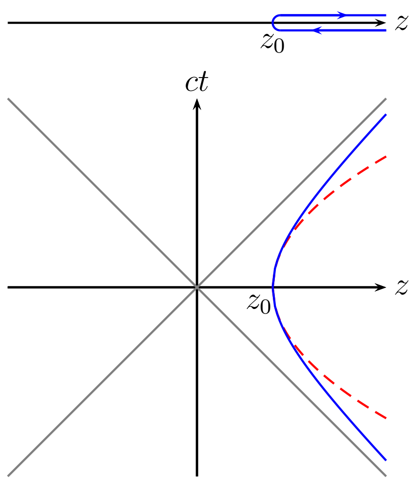

Figure 2: Problem 3

(a) This represents the world-line of a particle thrown from $z > z_0$ at $t < 0$ towards $z = z_0$ in region of constant (proper) acceleration α as described by the bold (blue) curve in the space-time diagram in Figure 3. In contrast a Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$z - z_0 = \frac{1}{2}\alpha t^2 \tag{4}$$

as described by the dashed (red) curve in the space-time diagram in Figure 3. Show that the hyperbolic curve

$$z = z_0\sqrt{1 + \frac{c^2 t^2}{z_0^2}} \tag{5}$$

in regions that satisfy

$$t \ll \frac{c}{\alpha} \tag{6}$$

is approximately the parabolic curve

$$z = z_0 + \frac{1}{2}\alpha t^2 + \dots . \tag{7}$$

(b) Recognize that the proper acceleration α does not have an upper bound.

(c) A large acceleration is achieved by taking an above turn while moving very fast. Thus, turning around while moving close to the speed of light c should achieve the highest acceleration. Show that $\alpha \to \infty$ corresponding to $z_0 \to 0$ represents this scenario. What is the equation of motion of a particle moving with infinite proper acceleration. To gain insight, plot world-lines of particles moving with $\alpha = c^2/z_0$, $\alpha = 10c^2/z_0$, and $\alpha = 100c^2/z_0$.

4. (**20 points.**) A relativisitic particle in a uniform magnetic field is described by the equations

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v}, \tag{8a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{8b}$$

where

$$E = mc^2\gamma, \tag{9a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{9b}$$

and

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \tag{10}$$

Show that

$$\frac{d\gamma}{dt} = 0. \tag{11}$$

Then, derive

$$\frac{d\mathbf{v}}{dt} = \mathbf{v} \times \boldsymbol{\omega}_c, \tag{12}$$

where

$$\boldsymbol{\omega}_c = \frac{q\mathbf{B}}{m\gamma}. \tag{13}$$

Compare this relativistic motion to the associated non-relativistic motion.

5. (**20 points.**) A relativisitic particle in a uniform electric field is described by the equations

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v}, \tag{14a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{14b}$$

where

$$E = mc^2\gamma, \tag{15a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{15b}$$

and

$$\mathbf{F} = q\mathbf{E}. \tag{16}$$

Let us consider the configuration with the electric field in the $\hat{\mathbf{y}}$ direction,

$$\mathbf{E} = E\,\hat{\mathbf{y}}, \tag{17}$$

and initial conditions

$$\mathbf{v}(0) = 0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{18a}$$

$$\mathbf{x}(0) = 0\,\hat{\mathbf{x}} + y_0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}. \tag{18b}$$

(a) In terms of the definition

$$\boldsymbol{\omega}_0 = \frac{1}{c}\frac{q\mathbf{E}}{m}, \tag{19}$$

show that the equations of motion are given by

$$\frac{d\gamma}{dt} = \boldsymbol{\omega}_0 \cdot \boldsymbol{\beta} \tag{20}$$

and

$$\frac{d}{dt}(\boldsymbol{\beta}\gamma) = \boldsymbol{\omega}_0. \tag{21}$$

(b) Since the particle starts from rest show that we have

$$\boldsymbol{\beta}\gamma = \boldsymbol{\omega}_0 t. \tag{22}$$

For our configuration this implies

$$\beta_x = 0, \tag{23a}$$
$$\beta_y \gamma = \omega_0 t, \tag{23b}$$
$$\beta_z = 0. \tag{23c}$$

Further, deduce

$$\beta_y = \frac{\omega_0 t}{\sqrt{1 + \omega_0^2 t^2}}. \tag{24}$$

Integrate again and use the initial condition to show that the motion is described by

$$y - y_0 = \frac{c}{\bar{\omega}_0}\left[\sqrt{1 + \bar{\omega}_0^2 t^2} - 1\right]. \tag{25}$$

Rewrite the solution in the form

$$\left(y - y_0 + \frac{c}{\omega_0}\right)^2 - c^2 t^2 = \frac{c^2}{\omega_0^2}. \tag{26}$$

This represents a hyperbola passing through $y = y_0$ at $t = 0$. If we choose the initial position $y_0 = c/\omega_0$ we have

$$y^2 - c^2 t^2 = y_0^2. \tag{27}$$

(c) The (constant) proper acceleration associated with this motion is

$$\alpha = \omega_0 c = \frac{c^2}{y_0}. \tag{28}$$

A Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$y - y_0 = \frac{1}{2}\alpha t^2. \tag{29}$$

Show that the hyperbolic curve

$$y = y_0 \sqrt{1 + \frac{c^2 t^2}{y_0^2}} \tag{30}$$

in regions that satisfy

$$\omega_0 t \ll 1 \tag{31}$$

is approximately the parabolic curve

$$y = y_0 + \frac{1}{2}\alpha t^2 + \dots . \tag{32}$$