

Final Exam (Fall 2024)

PHYS 500A: MATHEMATICAL METHODS

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1. (20 points.) The generating function for the spherical harmonics, $Y_{lm}(\theta, \phi)$, is

$$\frac{1}{l!} \left(\mathbf{a} \cdot \frac{\mathbf{r}}{r} \right)^l = \sum_{m=-l}^l \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta, \phi) \psi_{lm}, \quad (1)$$

where the left hand side is expressed in terms of

$$\mathbf{r} = r(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (2)$$

$$\mathbf{a} = \frac{1}{2}(y_-^2 - y_+^2, -iy_-^2 - iy_+^2, 2y_-y_+), \quad (3)$$

and the right hand side consists of

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}} \frac{y_-^{l-m}}{\sqrt{(l-m)!}} \quad (4)$$

and

$$Y_{lm}(\theta, \phi) = e^{im\phi} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l+m)!}{(l-m)!}} \frac{1}{(\sin \theta)^m} \left(\frac{d}{d \cos \theta} \right)^{l-m} \frac{(\cos^2 \theta - 1)^l}{2^l l!}. \quad (5)$$

An example of a null-vector is

$$\mathbf{a} = (-i \cos \alpha, -i \sin \alpha, 1). \quad (6)$$

(a) Identify the corresponding y_{\pm} in Eq. (3) to show that, now, ψ_{lm} in Eq. (1) is

$$\psi_{lm} = \frac{e^{-im(\alpha - \frac{\pi}{2})}}{\sqrt{(l+m)!(l-m)!}}. \quad (7)$$

(b) Then, integrate Eq. (1) to derive an integral representation for spherical harmonics,

$$\frac{1}{l!} \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{im\alpha} [\cos \theta - i \sin \theta \cos(\phi - \alpha)]^l = \sqrt{\frac{4\pi}{2l+1}} \frac{i^m Y_{lm}(\theta, \phi)}{\sqrt{(l+m)!(l-m)!}}. \quad (8)$$

(c) By setting $m = 0$ derive the corresponding integral representation for Legendre polynomial $P_l(\cos \theta)$:

$$\int_0^\pi \frac{d\alpha}{\pi} [\cos \theta - i \sin \theta \cos \alpha]^l = P_l(\cos \theta). \quad (9)$$

(d) Use the integral representation for $J_0(t)$,

$$J_0(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{it \cos \alpha}, \quad (10)$$

to show that

$$P_l(\cos \theta) = \left(\cos \theta - \sin \theta \frac{d}{dt} \right)^l J_0(t) \Big|_{t=0}. \quad (11)$$

Verify this for $l = 0, 1, 2$.

(e) Now let $\theta = x/l$ and, for fixed x , consider the limit $l \rightarrow \infty$, to obtain

$$\lim_{l \rightarrow \infty} P_l \left(\cos \frac{x}{l} \right) = J_0(x), \quad (12)$$

which is often used in the approximate form

$$\theta \ll 1, l \gg 1 : \quad P_l(\cos \theta) \sim J_0(l\theta). \quad (13)$$

(f) For what geometrical reason does one expect an asymptotic connection between spherical and cylindrical coordinate functions?

2. (20 points.) Given

$$\left(\frac{a}{r} + \frac{\partial}{\partial r} \right) \left(\frac{b}{r} + \frac{\partial}{\partial r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}. \quad (14)$$

Find the numbers a and b .

3. (20 points.) Polynomials $(\mathbf{a} \cdot \mathbf{r})^l$ of degree l satisfy the Laplacian when \mathbf{a} is a null-vector, that is,

$$(\mathbf{a} \cdot \mathbf{a}) = 0. \quad (15)$$

(a) Show that

$$\nabla^2 (\mathbf{a} \cdot \mathbf{r})^l = l(l-1) (\mathbf{a} \cdot \mathbf{r})^{(l-2)} (\mathbf{a} \cdot \mathbf{a}), \quad (16)$$

and conclude

$$\nabla^2 (\mathbf{a} \cdot \mathbf{r})^l = 0. \quad (17)$$

(b) Write the polynomial construction in the form

$$(\mathbf{a} \cdot \mathbf{r})^l = r^l (\mathbf{a} \cdot \hat{\mathbf{r}})^l. \quad (18)$$

Observe that $(\mathbf{a} \cdot \hat{\mathbf{r}})^l$ has no radial dependence. Thus, in this form, the radial and angular dependence is separated. Starting from the Laplacian in spherical polar coordinates,

$$\left[\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] (\mathbf{a} \cdot \mathbf{r})^l = 0, \quad (19)$$

deduce

$$\frac{r^l}{r^2} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] (\mathbf{a} \cdot \hat{\mathbf{r}})^l + (\mathbf{a} \cdot \hat{\mathbf{r}})^l \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} r^l = 0. \quad (20)$$

(c) Show that

$$\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} r^l = l(l+1) \frac{r^l}{r^2}. \quad (21)$$

Thus, derive the differential equation for the generating function

$$\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + l(l+1) \right] (\mathbf{a} \cdot \hat{\mathbf{r}})^l = 0. \quad (22)$$

(d) Use the generating function

$$\frac{(\mathbf{a} \cdot \hat{\mathbf{r}})^l}{l!} = \sum_{m=-l}^l \psi_{lm} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta, \phi) \quad (23)$$

written in terms of

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}} \frac{y_-^{l-m}}{\sqrt{(l-m)!}} \quad (24)$$

to derive the differential equation for spherical harmonics

$$\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + l(l+1) \right] Y_{lm}(\theta, \phi) = 0. \quad (25)$$

4. (20 points.) For a null-vector \mathbf{a} , that satisfies

$$\mathbf{a} \cdot \mathbf{a} = 0, \quad (26)$$

the polynomial $(\mathbf{a} \cdot \hat{\mathbf{r}})^l$ of degree l is the generating function of spherical harmonics $Y_{lm}(\theta, \phi)$. To derive the orthonormality properties of spherical harmonics let us consider the product of two generating functions, with null-vectors \mathbf{a} and \mathbf{a}^* , integrated over all the angles,

$$\int d\Omega (\mathbf{a}^* \cdot \hat{\mathbf{r}})^l (\mathbf{a} \cdot \hat{\mathbf{r}})^{l'}, \quad (27)$$

where

$$d\Omega = \sin \theta d\theta d\phi. \quad (28)$$

(a) After integration over the angles the product of the two generating functions is a scalar. Thus, it has to be constructed out of $(\mathbf{a} \cdot \mathbf{a})$, $(\mathbf{a}^* \cdot \mathbf{a}^*)$, and $(\mathbf{a}^* \cdot \mathbf{a})$. Since $(\mathbf{a} \cdot \mathbf{a}) = 0$ and $(\mathbf{a}^* \cdot \mathbf{a}^*) = 0$, the integral has to be constructed out of $(\mathbf{a}^* \cdot \mathbf{a})$. This is possible only if $l = l'$. Together, we conclude

$$\int d\Omega (\mathbf{a}^* \cdot \hat{\mathbf{r}})^l (\mathbf{a} \cdot \hat{\mathbf{r}})^{l'} = \delta_{ll'} (\mathbf{a}^* \cdot \mathbf{a})^l C_l, \quad (29)$$

in terms of arbitrary constant C_l .

(b) To determine C_l choose

$$\mathbf{a} = (1, i, 0). \quad (30)$$

For this choice of null-vector, evaluate $\mathbf{a}^* = (1, -i, 0)$, $(\mathbf{a} \cdot \hat{\mathbf{r}}) = \sin \theta e^{i\phi}$, $(\mathbf{a}^* \cdot \hat{\mathbf{r}}) = \sin \theta e^{-i\phi}$, and $(\mathbf{a}^* \cdot \hat{\mathbf{a}}) = 2$. Thus, find

$$C_l = \frac{4\pi}{2^l} \int_0^1 dt (1-t^2)^l, \quad (31)$$

after substituting $\cos \theta = t$. Evaluate

$$C_0 = 4\pi. \quad (32)$$

Integrate by parts in the integral for C_l to derive the recurrence relation

$$C_l = \frac{l}{2l+1} C_{l-1}. \quad (33)$$

Evaluate

$$C_l = \frac{4\pi 2^l l! l!}{(2l+1)!}. \quad (34)$$

Thus, conclude

$$\int d\Omega \frac{(\mathbf{a}^* \cdot \hat{\mathbf{r}})^l (\mathbf{a} \cdot \hat{\mathbf{r}})^{l'}}{l! l!} = \delta_{ll'} 4\pi \frac{(\mathbf{a}^* \cdot \mathbf{a})^l 2^l}{(2l+1)!}. \quad (35)$$

(c) For null-vectors constructed out of y_{\pm} in the form

$$\mathbf{a} = \left(\frac{y_-^2 - y_+^2}{2}, \frac{y_-^2 + y_+^2}{2i}, y_+ y_- \right) \quad (36)$$

show that

$$4\pi \frac{(\mathbf{a}^* \cdot \mathbf{a})^l 2^l}{(2l+1)!} = \frac{4\pi}{2l+1} \sum_{m=-l}^l \sum_{m'=-l'}^{l'} \psi_{lm}^* \psi_{l'm'} \delta_{mm'}, \quad (37)$$

where

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}} \frac{y_-^{l-m}}{\sqrt{(l-m)!}}. \quad (38)$$

Using the generating function

$$\frac{(\mathbf{a}^* \cdot \hat{\mathbf{r}})^l}{l!} = \sum_{m=-l}^l \psi_{lm} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta, \phi) \quad (39)$$

show that

$$\begin{aligned} & \sum_{m=-l}^l \sum_{m'=-l'}^{l'} \psi_{lm}^* \psi_{l'm'} \sqrt{\frac{4\pi}{2l+1}} \sqrt{\frac{4\pi}{2l'+1}} \int d\Omega Y_{lm}^*(\theta, \phi) Y_{l'm'}(\theta, \phi) \\ &= \delta_{ll'} \sqrt{\frac{4\pi}{2l+1}} \sqrt{\frac{4\pi}{2l'+1}} \sum_{m=-l}^l \sum_{m'=-l'}^{l'} \psi_{lm}^* \psi_{l'm'} \delta_{mm'}. \end{aligned} \quad (40)$$

Thus, comparing the two sides of the equality, read out the orthonormality condition for the spherical harmonics,

$$\int d\Omega Y_{lm}^*(\theta, \phi) Y_{l'm'}(\theta, \phi) = \delta_{ll'} \delta_{mm'}. \quad (41)$$

5. **(20 points.)** Check the orthonormality condition for the spherical harmonics

$$\int d\Omega Y_{lm}^*(\theta, \phi) Y_{l'm'}(\theta, \phi) = \delta_{ll'} \delta_{mm'} \quad (42)$$

for $l = 1, m = 1, l' = 2, m' = 1$, by explicitly evaluating the integral on the left hand side.