Midterm Exam No. 01 (Fall 2024)

PHYS 500A: MATHEMATICAL METHODS

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1. (**20 points.**) Consider the dyadic construction of an unitary operator

$$\mathbf{U} = \hat{\mathbf{i}}\,\hat{\mathbf{j}} + \hat{\mathbf{j}}\,\hat{\mathbf{i}}, \tag{1}$$

where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are orthonormal vectors satisfying the completeness relation

$$\mathbf{1} = \hat{\mathbf{i}}\,\hat{\mathbf{i}} + \hat{\mathbf{j}}\,\hat{\mathbf{j}}. \tag{2}$$

Evaluate

$$\mathrm{tr}(\mathbf{U}^{107}). \tag{3}$$

2. (**20 points.**) A uniformly charged infinitely thin wire of length a and total charge Q is placed on the z axis such that one end of the wire is at the origin. Write down the charge density of the wire in terms of δ-function(s) and Heaviside step function(s). Integrate the charge density over all space to verify that it indeed returns the total charge on the wire.

3. (**20 points.**) Transformation of basis vectors.

   (a) Let us consider a set of basis vectors $\mathbf{e}_i$, where $i = 1, 2, 3$, and the associated reciprocal basis vectors $\mathbf{e}^i$ that satisfy the completeness relation

$$\mathbf{1} = \mathbf{e}^i\,\mathbf{e}_i = \mathbf{e}^1\mathbf{e}_1 + \mathbf{e}^2\mathbf{e}_2 + \mathbf{e}^3\mathbf{e}_3. \tag{4}$$

   It is a complete set because an arbitrary vector $\mathbf{A}$ can be expressed in terms of it's projections along the basis vectors in the following way,

$$\mathbf{A} = \mathbf{A}\cdot\mathbf{1} = \mathbf{A}\cdot(\mathbf{e}^i\,\mathbf{e}_i) = (\mathbf{A}\cdot\mathbf{e}^i)\,\mathbf{e}_i = A^i\,\mathbf{e}_i, \tag{5}$$

   where we recognized and defined the projections of vector $\mathbf{A}$ along the direction of basis vectors as the components

$$A^i = (\mathbf{A}\cdot\mathbf{e}^i). \tag{6}$$

   Similarly, multiplying by the identity dyadic on the left gives

$$\mathbf{A} = \mathbf{1}\cdot\mathbf{A} = (\mathbf{e}^i\,\mathbf{e}_i)\cdot\mathbf{A} = \mathbf{e}^i\,(\mathbf{e}_i\cdot\mathbf{A}) = \mathbf{e}^i\,A_i, \tag{7}$$

   where now the projections of vector $\mathbf{A}$ in the direction of the reciprocal basis vectors are the components

$$A_i = (\mathbf{e}_i\cdot\mathbf{A}). \tag{8}$$

For consistency we require the equality

$$(\mathbf{A} \cdot \mathbf{e}^i)\, \mathbf{e}_i = \mathbf{e}^i\, (\mathbf{e}_i \cdot \mathbf{A}). \tag{9}$$

Thus, derive

$$A^j = g^{ji} A_i, \tag{10a}$$
$$A_j = g_{ji} A^i, \tag{10b}$$

where the metric tensors are defined as

$$g^{ji} = \mathbf{e}^j \cdot \mathbf{e}^i, \tag{11a}$$
$$g_{ji} = \mathbf{e}_j \cdot \mathbf{e}_i. \tag{11b}$$

(b) For another set of basis vectors $\mathbf{g}_i$ and the associated reciprocal basis vectors $\mathbf{g}^i$ that satisfy the completeness relation

$$\mathbf{1} = \mathbf{g}^i\, \mathbf{g}_i \tag{12}$$

we can write

$$\mathbf{A} = (\mathbf{A} \cdot \mathbf{g}^i)\, \mathbf{g}_i = \bar{A}^i\, \mathbf{g}_i, \tag{13}$$

where the components $\bar{A}^i$ are in general different from A^i, and

$$\mathbf{A} = \mathbf{g}^i\, (\mathbf{g}_i \cdot \mathbf{A}) = \mathbf{g}^i\, \bar{A}_i. \tag{14}$$

For consistency we require the equality

$$(\mathbf{A} \cdot \mathbf{g}^i)\, \mathbf{g}_i = \mathbf{g}^i\, (\mathbf{g}_i \cdot \mathbf{A}). \tag{15}$$

Thus, derive

$$\bar{A}^j = \bar{g}^{ji} \bar{A}_i, \tag{16a}$$
$$\bar{A}_j = \bar{g}_{ji} \bar{A}^i, \tag{16b}$$

where the metric tensors are defined as

$$\bar{g}^{ji} = \mathbf{g}^j \cdot \mathbf{g}^i, \tag{17a}$$
$$\bar{g}_{ji} = \mathbf{g}_j \cdot \mathbf{g}_i. \tag{17b}$$

(c) For consistency between the two independent basis vector representations we require the equality

$$(\mathbf{A} \cdot \mathbf{e}^i)\, \mathbf{e}_i = (\mathbf{A} \cdot \mathbf{g}^i)\, \mathbf{g}_i. \tag{18}$$

Taking the dot product on the right with $\mathbf{e}^j$ and using orthogonality relation

$$\mathbf{e}_i \cdot \mathbf{e}^i = \delta^i_j \tag{19}$$

we obtain

$$(\mathbf{A} \cdot \mathbf{e}^j) = (\mathbf{A} \cdot \mathbf{g}^i) [\mathbf{g}_i \cdot \mathbf{e}^j]. \tag{20}$$

Similarly, taking the dot product on the right with $\mathbf{g}^j$ we obtain

$$(\mathbf{A} \cdot \mathbf{e}^i) [\mathbf{e}_i \cdot \mathbf{g}^j] = (\mathbf{A} \cdot \mathbf{g}^j). \tag{21}$$

In terms of the transformation matrices connecting the two basis vectors,

$$S_i{}^j = [\mathbf{g}_i \cdot \mathbf{e}^j] \tag{22}$$

and

$$R_i{}^j = [\mathbf{e}_i \cdot \mathbf{g}^j], \tag{23}$$

we can derive the transformation of the components

$$A^j = \bar{A}^i S_i{}^j, \tag{24a}$$

$$A^i R_i{}^j = \bar{A}^j. \tag{24b}$$

Further, derive

$$A_j = R_j{}^i \bar{A}_i, \tag{25a}$$

$$S_j{}^i A_i = \bar{A}^j. \tag{25b}$$

Show that

$$R_i{}^j S_j{}^k = [\mathbf{e}_i \cdot \mathbf{g}^j][\mathbf{g}_j \cdot \mathbf{e}^k] = \mathbf{e}_i \cdot (\mathbf{g}^j \, \mathbf{g}_j) \cdot \mathbf{e}^k = \mathbf{e}_i \cdot \mathbf{1} \cdot \mathbf{e}^k = \delta_i^k. \tag{26}$$

(d) i. Find the transformation matrices S and T between cylindrical polar coordinates and rectangular coordinates. Verify that $ST = 1$.

ii. Find the transformation matrices S and T between cylindrical polar coordinates and spherical polar coordinates. Verify that $ST = 1$.

4. (**20 points.**) Find the cube roots of unity by solving the equation

$$z^3 = 1, \tag{27}$$

where the exponent of z is a positive integer.

(a) Find the roots of the equation

$$z^{\frac{3}{2}} = 1, \tag{28}$$

where the exponent of z is a rational number.

(b) Find the roots of the equation

$$z^{\pi} = 1, \tag{29}$$

where the exponent of z is an irrational number.

5. (**20 points.**) Check if the function

$$f(z) = \ln \frac{(z-1)}{(z+1)} \tag{30}$$

satisfies the Cauchy-Riemann conditions. Investigate the geometric properties of this function using ComplexContourPlot in Mathematica.