Homework No. 02 (Fall 2024)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Friday, 2023 Sep 6, 4.30pm

1. (**20 points.**) Verify the following identities:

$$\boldsymbol{\nabla} r = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}, \tag{1a}$$

$$\boldsymbol{\nabla}\, \mathbf{r} = \mathbf{1}. \tag{1b}$$

Further, show that

$$\boldsymbol{\nabla} \cdot \mathbf{r} = 3, \tag{2a}$$

$$\boldsymbol{\nabla} \times \mathbf{r} = 0. \tag{2b}$$

Here r is the magnitude of the position vector $\mathbf{r}$, and $\hat{\mathbf{r}}$ is the unit vector pointing in the direction of $\mathbf{r}$.

2. (**20 points.**) Evaluate the left hand side of the equation

$$\boldsymbol{\nabla}(\mathbf{r} \cdot \mathbf{p}) = a\,\mathbf{p} + b\,\mathbf{r}, \tag{3}$$

where $\mathbf{p}$ is a constant vector. Thus, find a and b.

3. (**20 points.**) Evaluate

$$\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{r}}{r^3}\right), \tag{4}$$

everywhere in space, including $\mathbf{r} = 0$.
Hint: Check your answer for consistency by using divergence theorem.

4. (**20 points.**) Evaluate

$$\boldsymbol{\nabla} \left(\frac{\mathbf{p} \cdot \mathbf{r}}{r^3}\right), \tag{5}$$

where $\mathbf{p}$ is a constant vector.

5. (**20 points.**) Consider the distribution

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}. \tag{6}$$

Show that

$$\delta(x) \begin{cases} \to \infty, & \text{if} \quad x = 0, \\ \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{7}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{8}$$

Plot $\delta(x)$ before taking the limit $\varepsilon \to 0$ and identify ε in the plot.

6. (**10 points.**) A uniformly charged infinitely thin disc of radius R and total charge Q is placed on the x-y plane such that the normal vector is along the z axis and the center of the disc at the origin. Write down the charge density of the disc in terms of δ-function(s). Integrate over the charge density and verify that it returns the total charge on the disc.

7. (**10 points.**) An (idealized) infinitely long wire, (on the z-axis with infinitesimally small cross sectional area,) carrying a current I can be mathematically represented by the current density

$$\mathbf{J}(\mathbf{x}) = \hat{\mathbf{z}}\, I\, \delta(x)\delta(y). \tag{9}$$

A similar idealized wire forms a circular loop and is placed on the xy-plane with the center of the circular loop at the origin. Write down the current density of the circular loop carrying current I.