Homework No. 04 (Fall 2024)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2024 Sep 23, 4.30pm

1. (**20 points.**) For a given complex number z, say

$$z = \sqrt{2}\, e^{i\frac{\pi}{3}}, \tag{1}$$

evaluate

$$z^2, z^3, z^4, z^5, z^6, z^7, z^8, z^9, z^{10}. \tag{2}$$

Mark all of them on the complex plane. Decipher the pattern.

2. (**20 points.**) Evaluate

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{23}. \tag{3}$$

Mark the resulting number on the complex plane.

3. (**20 points.**) Prove the identity

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}. \tag{4}$$

Use the identity

$$(2+i)(3+i) = 5 + i5. \tag{5}$$

Similarly, find y/x in the relation

$$\tan^{-1}\left(\frac{3}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{y}{x}\right). \tag{6}$$

4. (**20 points.**) Verify that

$$\sqrt{-2}\sqrt{-3} = -\sqrt{6}. \tag{7}$$

However, it is often tempting to conclude

$$\sqrt{-2}\sqrt{-3} = \sqrt{(-2)(-3)} = \sqrt{6}. \tag{8}$$

The ambiguity in the interpretation of $\sqrt{-2}$ and $\sqrt{-3}$ is (partly) removed by writing,

$$\sqrt{-2} = \left(2e^{i\pi}\right)^{\frac{1}{2}} = \sqrt{2}e^{i\frac{\pi}{2}}, \tag{9a}$$

$$\sqrt{-3} = \left(3e^{i\pi}\right)^{\frac{1}{2}} = \sqrt{3}e^{i\frac{\pi}{2}}. \tag{9b}$$

This only partly removes the ambiguity because $\sqrt{-2}$ and $\sqrt{-3}$ have two independent roots each and Eqs. (9) only identifies one of the roots, the principal root, for each. Using Eqs. (9) verify the correctness of the statement in Eq. (7) again. The above ambiguity in the interpretation and the related confusions plagued the development of ideas related to complex numbers until the geometric visualization of a complex number using Argand diagram (magnitude and direction in polar representation) was discovered by Wessel in 1797 and popularized by Argand in 1806. Without this geometric interpretation even Euler fell into the trap of concluding $\sqrt{-2}\sqrt{-3} = \sqrt{6}$. So, is the statement in Eq. (8) erroneous? No. To this end, let us remove the ambiguity completely by recognizing the multiplicities in the roots,

$$\sqrt{-2} = \left(2e^{i\pi}\right)^{\frac{1}{2}} = \sqrt{2}e^{i\frac{\pi}{2}}(1,\omega), \quad \omega = e^{i\pi}, \tag{10a}$$

$$\sqrt{-3} = \left(3e^{i\pi}\right)^{\frac{1}{2}} = \sqrt{3}e^{i\frac{\pi}{2}}(1,\omega), \quad \omega = e^{i\pi}, \tag{10b}$$

where comma-separated quantities contribute to multiplicities in roots. Muliplication of the two roots of $\sqrt{-2}$ and two roots of $\sqrt{-3}$ leads to four possibilities,

$$(1,\omega) \times (1,\omega) \to (1,\omega,\omega,\omega^2). \tag{11}$$

Using $\omega^2 = 1$, only two out of four possibilities are independent. Thus, we have

$$\sqrt{-2}\sqrt{-3} = \sqrt{2}\sqrt{3}(1,\omega), \quad \omega = e^{i\pi}. \tag{12}$$

In summary, both the statements in Eqs. (7) and (8) are correct.

5. (**20 points.**) The close connection between the geometry of a complex number

$$z = x + iy \tag{13}$$

and a two-dimensional vector

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} \tag{14}$$

is intriguing. They have the same rules for addition and subtraction, but differ in their rules for multiplication. Show that

$$z_1^* z_2 = (\mathbf{r}_1 \cdot \mathbf{r}_2) + i(\mathbf{r}_1 \times \mathbf{r}_2) \cdot \hat{\mathbf{k}}. \tag{15}$$

In the quest for a number system that corresponds to a three dimensional vector, Hamilton in 1843 invented the quaternions. A quaternion P can be expressed in terms of Pauli matrices as

$$P = a_0 - i\mathbf{a} \cdot \boldsymbol{\sigma}. \tag{16}$$

Recall that the Pauli matrices are completely characterized by the identity

$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) = (\mathbf{a} \cdot \mathbf{b}) + i(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{17}$$

(a) Show that the (Hamilton) product of two quaternions,

$$P = a_0 - i\mathbf{a} \cdot \boldsymbol{\sigma}, \tag{18a}$$

$$Q = b_0 - i\mathbf{b} \cdot \boldsymbol{\sigma}, \tag{18b}$$

is given by

$$PQ = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) - i(a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{19}$$

(b) Verify that the Hamilton product is non-commutative. Determine $\big[P, Q\big]$.
Solution:

$$\big[P, Q\big] = -2i(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{20}$$

6. (**20 points.**) Find the fifth roots of unity by solving the equation

$$z^5 = 1. \tag{21}$$

Mark the points corresponding to the five roots on the complex plane. Find the five roots of the equation

$$z^5 = -1. \tag{22}$$

Mark the roots on the complex plane. Next, find the roots of the equation

$$z^5 = i \tag{23}$$

and mark the roots on the complex plane. Repeat the exercise for $z^5 = -i$. How do these roots match with the fifth roots of unity? Recognize the pattern.

7. (**20 points.**) Locate $z = \pi^i$ on the complex plane.