Homework No. 05 (Fall 2024)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2024 Sep 30, 4.30pm

1. (**20 points.**) Recall that analytic functions satisfy the Cauchy-Riemann equations. That is, the real and imaginary parts of an analytic function

$$f(x+iy) = u(x,y) + iv(x,y) \tag{1}$$

satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{2a}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{2b}$$

Given $f(z)$ and $g(z)$ are analytic functions in a region, then show that $f(g(z))$ satisfies the Cauchy-Riemann equations there.

Hint: Let $g = u + iv$ and $f = U + iV$. Thus, we can write

$$f(g(z)) = U(u(x,y), v(x,y)) + iV(u(x,y), v(x,y)). \tag{3}$$

2. (**20 points.**) Given an analytic function

$$f(z) = u(x,y) + iv(x,y) \tag{4}$$

on a complex plane, $z = x + iy$, we can imagine the two functions $u(x,y)$ and $v(x,y)$ to exist on the two-dimensional plane spanned by the real variables x and y. (Recall that even though addition and subtraction are identical in these spaces, the algebra of multiplication is different. Division is not introduced in a vector space.) In terms of the gradient operator in the two-dimensional vector space,

$$\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y}, \tag{5}$$

show that Cauchy-Riemann equations for $u(x,y)$ and $v(x,y)$ imply

$$(\boldsymbol{\nabla} u)\cdot(\boldsymbol{\nabla} v) = 0. \tag{6}$$

Thus, interpret that u's and v's are orthogonal family of surfaces and thus serves as a suitable chart for coordinatization. Mathematica allows visualization of these surfaces using the command ComplexContourPlot.

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f[z_] = z^3;
ComplexContourPlot[ReIm[f[z]], {z,-3-3 I,3+3 I}]
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The above two-line code in Mathematica plots the real and imaginary surfaces associated with the analytic function $f(z) = z^3$ between the coordinate points $(-3, -3)$ and $(3, 3)$.

3. (**20 points.**) Analytic functions are significantly constrained, in that they have to satisfy the Cauchy-Riemann conditions. These conditions are necessary (but not sufficient) for a function of a complex variable to be analytic (differentiable). Check if the following functions satisfy the Cauchy-Riemann conditions. If $f(z)$ is analytic for all z, then report the derivative as a function of z. Otherwise, determine the points, or regions, in the z plane where the function is not analytic.

$$f(z) = z^3, \tag{7a}$$
$$f(z) = |z|^2, \tag{7b}$$
$$f(z) = e^{iz}, \tag{7c}$$
$$f(z) = \ln z, \tag{7d}$$
$$f(z) = e^z + e^{iz}. \tag{7e}$$

Use ComplexContourPlot in Mathematica to visualize these functions.

4. (**20 points.**) Check if the function

$$f(z) = zz^* \tag{8}$$

satisfies the Cauchy-Riemann conditions.

(a) Verify that all the points for $f(z)$ lies on the non-negative real line.

(b) Verify that as you approach the point $z = r \geq 0$ on the non-negative real line, along a circle of fixed radius r from the frist quadrant, we have

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\theta \to 0} \frac{f(re^{i\theta}) - f(r)}{re^{i\theta} - r} = 0. \tag{9}$$

Then verify that as you approach the point $z = r$ along the real axis we have

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 + y^2 - (x^2 + y^2)}{\Delta x} = 2x. \tag{10}$$

(c) Thus, conclude that the derivative is not isotropic for any z.

(d) Use ComplexContourPlot in Mathematica to visualize these functions. Note that this is not an analytic function.