Homework No. 06 (Fall 2024)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Friday, 2024 Oct 11, 4.30pm

1. (**20 points.**) Evaluate the contour integral

$$I = \frac{1}{2\pi i} \oint_c dz \frac{e^{iz}}{(z^2 - a^2)}, \tag{1}$$

where the contour c is a unit circle going counterclockwise with center at the origin. Inquire the cases when $|a| > 1$ and $|a| < 1$.

2. (**20 points.**) Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx\, e^{iax}}{x^2 + 1} \tag{2}$$

using Cauchy's theorem, after choosing a suitable contour. Here a is real.

3. (**20 points.**) Consider the integral

$$I(a) = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{1}{1 - 2a\cos\theta + a^2}, \tag{3}$$

where a is complex.

   (a) Substitute $z = e^{i\theta}$, such that

$$2\cos\theta = z + \frac{1}{z}, \tag{4}$$

   and express the integral as a contour integral along the unit circle going counterclockwise. Locate the poles.

   (b) Evaluate the residues and show that

$$I(a) = \begin{cases} \dfrac{1}{1-a^2}, & \text{if } |a| < 1, \\ \dfrac{1}{a^2-1}, & \text{if } |a| > 1. \end{cases} \tag{5}$$

   (c) Plot $I(a)$ for real values of a. Plot real and imaginary part of $I(a)$ for complex a. Argue that $I(1)$ is divergent.

4. (**20 points.**) The following lecture recording from Fall 2020 available at

motivates the idea of analytic continuation. Let us consider the function

$$\mu(s) = \frac{1}{s}, \qquad s \neq 0. \tag{6}$$

(a) An integral representation of the function is

$$\mu(s) = \int_0^1 dt\, t^{s-1}, \qquad \mathrm{Re}(s) > 0. \tag{7}$$

Evaluate the integral and show that the integral is indeed equal to $1/s$ for $\mathrm{Re}(s) > 0$. However, the above integral representation breaks down for $\mathrm{Re}(s) \leq 0$. Show that

$$\mu(0) = \int_0^1 \frac{dt}{t} = \lim_{\delta\to 0} \int_\delta^1 \frac{dt}{t} = -\lim_{\delta\to 0} \ln \delta \tag{8}$$

is logarithmically divergent. Similarly, show that

$$\mu(-1) = \int_0^1 \frac{dt}{t^2} = \lim_{\delta\to 0} \int_\delta^1 \frac{dt}{t^2} = \lim_{\delta\to 0} \left[1 - \frac{1}{\delta}\right] \frac{1}{(-1)} \tag{9}$$

is divergent. Check out $\mu(-2)$.

(b) Another representation of the function valid on the complete complex plane of s is

$$\mu(s) = \frac{1}{(e^{i2\pi s} - 1)} \int_c dz\, z^{s-1}, \quad s \neq 0, \tag{10}$$

where the integral is evaluated on the contour $c = c_1 + c_2 + c_3$ described in Figure 1. Since the integral representation in Eq. (10) does not have the restriction $\mathrm{Re}(s) > 0$, and because its values are identical to the integral representation in Eq. (7) for $\mathrm{Re}(s) > 0$, it is the analytic continuation of the integral representation in Eq. (7).

i. For contour c_1 substitute $z = x\, e^{i\delta} \sim x + ix\delta$ and show that

$$\int_{c_1} dz\, z^{s-1} = \frac{1}{s}\Big(\epsilon^s - 1\Big). \tag{11}$$

ii. For contour c_3 substitute $z = \epsilon\, e^{i\theta}$ and show that

$$\int_{c_2} dz\, z^{s-1} = \frac{1}{s}\Big(e^{i2\pi s} - 1\Big)\epsilon^s. \tag{12}$$

iii. For contour c_3 substitute $z = x\, e^{i(2\pi-\delta)}$ and show that

$$\int_{c_3} dz\, z^{s-1} = \frac{1}{s}\Big(1 - \epsilon^s\Big)e^{i2\pi s}. \tag{13}$$

Together, we have

$$\mu(s) = \frac{1}{(e^{i2\pi s} - 1)} \frac{1}{s} \left[\Big(\epsilon^s - 1\Big) + \Big(e^{i2\pi s} - 1\Big)\epsilon^s + \Big(1 - \epsilon^s\Big)e^{i2\pi s}\right] = \frac{1}{s}. \tag{14}$$

Observe that the apparent divergence when the factor $(e^{i2\pi s} - 1)$ equals 0 for integer s is nonexistent.

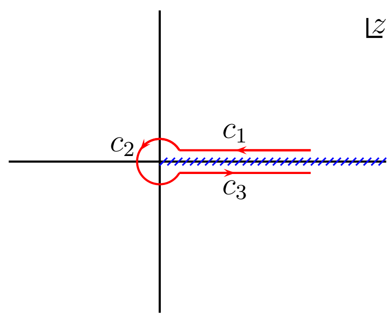

Figure 1: Contour $c = c_1 + c_2 + c_3$. The radius of the contour c_2 is ϵ and contours c_1 and c_3 are δ away from the real line. We assume limits $\epsilon \to 0$ and $\delta \to 0$.