Homework No. 07 (Fall 2024)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2024 Oct 28, 4.30pm

- (**Notation.**) The Fourier space is spanned by the Fourier eigenfunctions

$$e^{im\phi}, \qquad m = 0, \pm 1, \pm 2, \ldots, \qquad 0 \leq \phi < 2\pi. \tag{1}$$

An arbitrary function $f(\phi)$ has the Fourier series representation

$$f(\phi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} a_m e^{im\phi}, \tag{2}$$

where $e^{im\phi}$ are the Fourier eigenfunctions and a_m are the respective Fourier components.

1. (**20 points.**) Determine all the Fourier components a_m for the following functions: $\cos\phi$, $\sin\phi$, $\cos^2\phi$, $\sin^2\phi$, $\cos^3\phi$, $\sin^3\phi$.

2. (**20 points.**) Determine the particular function $f(\phi)$ that has the Fourier components

$$a_m = 1 \tag{3}$$

for all m. That is, all the Fourier coefficients are contributing equally in the series.

3. (**20 points.**) To determine the Fourier components of $\tan\phi$ start from

$$\tan\phi = \frac{1}{i} \frac{e^{i\phi} - e^{-i\phi}}{e^{i\phi} + e^{-i\phi}} \tag{4}$$

and show that

$$\tan\phi = \frac{1}{i} + \sum_{m=1}^{\infty} e^{-2im\phi} \frac{2(-1)^m}{i}. \tag{5}$$

Thus, read out all the Fourier components. Similarly, find the Fourier components of $\cot\phi$.

- (**Notation.**) The (continuous) Fourier space is spanned by the Fourier eigenfunctions

$$e^{ikx}, \qquad -\infty < k < \infty, \qquad -\infty < x < \infty. \tag{6}$$

An arbitrary function $f(x)$ has the Fourier series representation

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k), \tag{7}$$

where e^{ikx} are the Fourier eigenfunctions and $\tilde{f}(k)$ are the respective Fourier components.

4. (**20 points.**) Find the Fourier transform of a Gaussian function

$$f(x) = e^{-ax^2}. \tag{8}$$

That is, evalaute the integral

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} e^{-ax^2}. \tag{9}$$

5. (**20 points.**) The Heaviside step function is defined as

$$\theta(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases} \tag{10}$$

The Fourier transform and the corresponding inverse ae,

$$\theta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\tilde{\theta}(\omega), \tag{11a}$$

$$\tilde{\theta}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\theta(t). \tag{11b}$$

(a) Using the definition in Eq. (10) in Eq. (11b) show that

$$\tilde{\theta}(\omega) = \int_0^{\infty} dt\, e^{i\omega t} = \lim_{\delta\to 0+} \int_0^{\infty} dt\, e^{i\omega t} e^{-\delta t} = \lim_{\delta\to 0+} -\frac{1}{i}\frac{1}{\omega + i\delta}. \tag{12}$$

(b) Verify that

$$\theta(t) = \lim_{\delta\to 0+} -\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega\, \frac{e^{-i\omega t}}{\omega + i\delta} \tag{13}$$

is indeed an integral representation of Heaviside step function.

6. (**20 points.**) Consider the inhomogeneous linear differential equation

$$\left(a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \right) f(x) = \delta(x). \tag{14}$$

Use the Fourier transformation and the associated inverse Fourier transformation

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k), \tag{15a}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x), \tag{15b}$$

to show that the corresponding equation satisfied by $\tilde{f}(k)$ is algebraic. Find $\tilde{f}(k)$.

- (**Notation.**) The half-range Fourier space is spanned by the Fourier eigenfunctions

$$\sin m\phi, \qquad m = 1, 2, 3, \ldots, \qquad 0 \le \phi \le \pi. \tag{16}$$

An arbitrary function $f(\phi)$, for ϕ limited to half the range, has the half-range Fourier series representation

$$f(\phi) = \sum_{m=1}^{\infty} a_m \sin m\phi, \tag{17}$$

where $\sin m\phi$ are the half-range Fourier eigenfunctions and a_m are the respective half-range Fourier components.

7. (**20 points.**) For ϕ limited to the range

$$0 \le \phi \le \pi \tag{18}$$

show that $\cos\phi$ can be expressed as a linear combination of sin functions. That is,

$$\cos\phi = \sum_{m=1}^{\infty} a_m \sin m\phi. \tag{19}$$

Show that

$$a_m = \begin{cases} 0, & m = 1, 3, 5, \ldots, \\ \dfrac{4}{\pi} \dfrac{m}{(m^2 - 1)}, & m = 2, 4, 6, \ldots. \end{cases} \tag{20}$$

Note that the series expansion is not valid at the boundaries $\phi = 0$ and $\phi = \pi$.