Homework No. 08 (Fall 2024)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2024 Nov 4, 4.30pm

0. Following lectures from Fall 2023 are useful resources.

Damped harmonic oscillator: https://youtu.be/ANWBw6uxT6Q

Forced harmonic oscillator: https://youtu.be/uKYOyV3GS7g

1. (**20 points.**) A damped harmonic oscillator, constituting of a body of mass m and a spring of spring constant k, is described by

$$ma = -kx - bv, \tag{1}$$

where x is position, $v = dx/dt$ is velocity, $a = dv/dt$ is acceleration, and b is the damping coefficient. Thus, we have the differential equation

$$\left[\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right] x(t) = 0 \tag{2}$$

with initial conditions

$$x(0) = x_0, \tag{3a}$$

$$\dot{x}(0) = v_0, \tag{3b}$$

where

$$\omega_0^2 = \frac{k}{m}, \qquad 2\gamma = \frac{b}{m}. \tag{4}$$

(a) $\gamma = 0$: In the absence of damping show that the solution is

$$x(t) = x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t. \tag{5}$$

(b) $\gamma < \omega_0$: Underdamped harmonic oscillator.

$$x(t) = e^{-\gamma t}\left[x_0\cos\sqrt{\omega_0^2-\gamma^2}\,t + \frac{(v_0+\gamma x_0)}{\sqrt{\omega_0^2-\gamma^2}}\sin\sqrt{\omega_0^2-\gamma^2}\,t\right]. \tag{6}$$

(c) $\gamma = \omega_0$: Critically damped harmonic oscillator.

$$x(t) = e^{-\omega_0 t}\left[x_0 + (v_0+\omega_0 x_0)t\right]. \tag{7}$$

(d) $\gamma > \omega_0$: Overdamped harmonic oscillator.

$$x(t) = e^{-\gamma t}\left[x_0 \cosh\sqrt{\gamma^2-\omega_0^2}t + \frac{(v_0+\gamma x_0)}{\sqrt{\gamma^2-\omega_0^2}}\sinh\sqrt{\gamma^2-\omega_0^2}t\right]. \tag{8}$$

(e) Set $\omega_0 = 1$, which is equivalent to the substitution $\omega_0 t = \tau$, and sets the scale for the time t. That is, time is measured in units of $T = 2\pi/\omega_0$. The system is then completely characterized by the parameter γ/ω_0 and the initial conditions x_0 and v_0. Plot the solutions for the initial conditions $x_0 = 0$ and $v_0 = 1$.

2. (**20 points.**) Starting from the solution for the position of an underdamped harmonic oscillator ($\gamma < \omega_0$),

$$x(t) = e^{-\gamma t}\left[x_0 \cos\sqrt{\omega_0^2-\gamma^2}t + \frac{(v_0+\gamma x_0)}{\sqrt{\omega_0^2-\gamma^2}}\sin\sqrt{\omega_0^2-\gamma^2}t\right], \tag{9}$$

obtain the solution for the velocity $v(t) = dx/dt$ of an underdamped harmonic oscillator ($\gamma < \omega_0$) in the form

$$v(t) = e^{-\gamma t}\left[v_0 \cos\sqrt{\omega_0^2-\gamma^2}t - \frac{(\omega_0^2 x_0+\gamma v_0)}{\sqrt{\omega_0^2-\gamma^2}}\sin\sqrt{\omega_0^2-\gamma^2}t\right]. \tag{10}$$

3. (**20 points.**) A critically damped harmonic oscillator is described by the differential equation

$$\left[\frac{d^2}{dt^2} + 2\omega_0\frac{d}{dt} + \omega_0^2\right]x(t) = 0, \tag{11}$$

where ω_0 is a characteristic frequency. Find the solution $x(t)$ for initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. Plot $x(t)$ as a function of t in the following graph where $x_0\, e^{-\omega_0 t}$ is already plotted for reference. For what t is the solution $x(t)$ a maximum?

4. (**20 points.**) A critically damped harmonic oscillator is described by the differential equation

$$\left[\frac{d^2}{dt^2} + 2\omega_0\frac{d}{dt} + \omega_0^2\right]x(t) = 0, \tag{12}$$

where ω_0 is a characteristic frequency. Find the solution $x(t)$ for initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$. Plot $x(t)$ as a function of t in the graph in Figure 2, where ω_0 and v_0/ω_0 is used to set scales for time t and position $x(t)$. For what t is the solution $x(t)$ a maximum?

5. (**20 points.**) A body experiencing only damping is described by the differential equation

$$\left[\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt}\right]x(t) = 0, \tag{13}$$

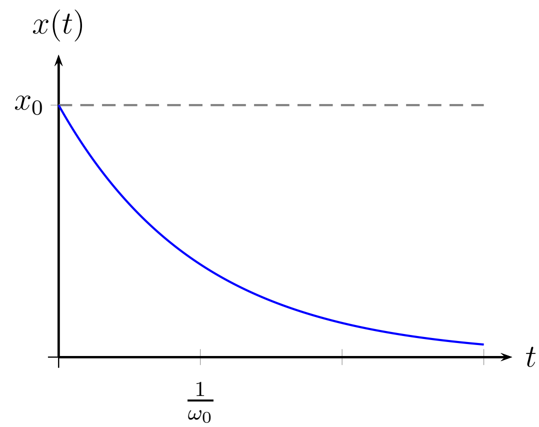

Figure 1: Critically damped harmonic oscillator.

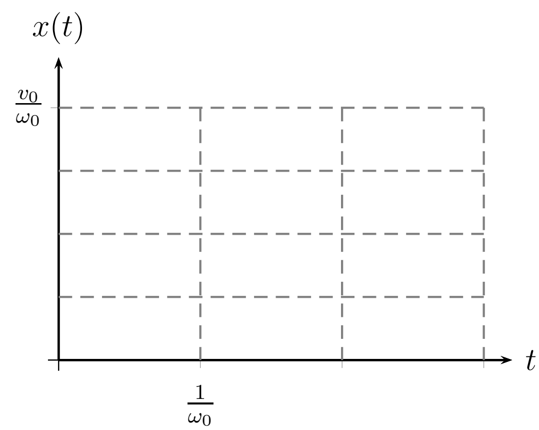

Figure 2: Critically damped harmonic oscillator.

where γ is a measure of the damping. Find the solution $x(t)$ for initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$ to be

$$x(t) = x_0 + \frac{v_0}{2\gamma}\left[1 - e^{-2\gamma t}\right]. \tag{14}$$

Obtain the above expression starting from the solution for the overdamped harmonic oscillator ($\gamma > \omega_0$)

$$x(t) = e^{-\gamma t}\left[x_0 \cosh\sqrt{\gamma^2 - \omega_0^2}\,t + \frac{(v_0 + \gamma x_0)}{\sqrt{\gamma^2 - \omega_0^2}} \sinh\sqrt{\gamma^2 - \omega_0^2}\,t\right] \tag{15}$$

by setting $\omega_0 = 0$. Interpret the solution for $v_0 = 0$, why isn't there no motion?

6. (**20 points.**) Starting from the solution for the underdamped harmonic oscillator ($\gamma < \omega_0$),

$$x(t) = e^{-\gamma t}\left[x_0 \cos\sqrt{\omega_0^2 - \gamma^2}\,t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin\sqrt{\omega_0^2 - \gamma^2}\,t\right], \tag{16}$$

obtain the solution for the overdamped harmonic oscillator ($\gamma > \omega_0$),

$$x(t) = e^{-\gamma t} \left[x_0 \cosh \sqrt{\gamma^2 - \omega_0^2}\, t + \frac{(v_0 + \gamma x_0)}{\sqrt{\gamma^2 - \omega_0^2}} \sinh \sqrt{\gamma^2 - \omega_0^2}\, t \right]. \tag{17}$$

7. (**20 points.**) Starting from the solution for the underdamped harmonic oscillator ($\gamma < \omega_0$),

$$x(t) = e^{-\gamma t} \left[x_0 \cos \sqrt{\omega_0^2 - \gamma^2}\, t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin \sqrt{\omega_0^2 - \gamma^2}\, t \right], \tag{18}$$

obtain the solution for the critically damped harmonic oscillator ($\gamma = \omega_0$),

$$x(t) = e^{-\omega_0 t} \left[x_0 + (v_0 + \omega_0 x_0) t \right]. \tag{19}$$

8. (**20 points.**) The solution for the underdamped harmonic oscillator ($\gamma < \omega_0$) is

$$x(t) = e^{-\gamma t} \left[x_0 \cos \sqrt{\omega_0^2 - \gamma^2}\, t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin \sqrt{\omega_0^2 - \gamma^2}\, t \right]. \tag{20}$$

For the initial condition $x_0 = 0$ we have

$$x(t) = \frac{v_0\, e^{-\gamma t}}{\sqrt{\omega_0^2 - \gamma^2}} \sin \sqrt{\omega_0^2 - \gamma^2}\, t. \tag{21}$$

Verify that the function

$$\frac{v_0\, e^{-\gamma t}}{\sqrt{\omega_0^2 - \gamma^2}} \tag{22}$$

is an envelope to the solution $x(t)$. Investigate if this is an envelope for the case $x_0 \neq 0$.

9. (**20 points.**) Read the article titled 'Life at low Reynolds number' by E. M. Purcell, American Journal of Physics 45 (1977) 3. Here is the link to the article:

http://dx.doi.org/10.1119/1.10903

Here is a question asked to verify the understanding of the concept being discussed in the paper. Imagine a micrometer sized bacteria, shaped like a human, swimming in water using the methods used by a typical human swimmer. Qualitatively describe the motion of this hypothetical bacteria.