Homework No. 09 (Fall 2024)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Friday, 2024 Nov 15, 4.30pm

0. Following lectures from Fall 2023 are useful resources.

https://youtu.be/wDuH2X3Rxig	Green's function, particular solution
https://youtu.be/5ffT75Aae8Q	Green's function and continuity conditions
https://youtu.be/WetQo3tnuq8	Retarded versus advanced Green's function
https://youtu.be/oHCnYqDkDfk	Green's function and boundary conditions

1. (**20 points.**) A forced harmonic oscillator, in the absence of damping, constituting of a body of mass m and a spring of spring constant k, is described by

$$ma + kx = F(t), \tag{1}$$

where x is position, $v = dx/dt$ is velocity, $a = dv/dt$ is acceleration, and $F(t)$ is a driving force. Thus, we have the differential equation

$$-\left[\frac{d^2}{dt^2} + \omega_0^2\right] x(t) = A(t), \tag{2}$$

where

$$\omega_0^2 = \frac{k}{m}, \qquad A(t) = -\frac{F(t)}{m}. \tag{3}$$

Let us consider the case with initial conditions

$$x(0) = 0, \tag{4a}$$
$$\dot{x}(0) = 0. \tag{4b}$$

Verify by substitution that

$$x(t) = -\frac{1}{\omega_0}\int_0^t dt' \sin\omega_0(t-t')A(t') \tag{5}$$

is the solution.

2. (**20 points.**) Consider the differential equation

$$-\left[\frac{d^2}{dt^2} + \omega_0^2\right] x(t) = -\omega_f^2 x_f \sin\omega_f t, \tag{6}$$

with initial conditions

$$x(0) = 0, \tag{7a}$$
$$\dot{x}(0) = 0. \tag{7b}$$

Verify by substitution that

$$x(t) = -x_f \frac{\omega_f^2}{(\omega_0^2 - \omega_f^2)} \frac{1}{\omega_0} \Big[\omega_f \sin \omega_0 t - \omega_0 \sin \omega_f t \Big] \tag{8}$$

is the solution. Show that the first term in the solution, called the transient solution, is solution to the homogeneous part of the differential equation. Show that the second term in the solution, called the steady-state solution, is a particular solution to the inhomogeneous differential equation.

3. (**20 points.**) Verify that
$$\frac{d}{dz}|z| = \theta(z) - \theta(-z), \tag{9}$$
where $\theta(z) = 1$, if $z > 0$, and 0, if $z < 0$. Further, verify that
$$\frac{d^2}{dz^2}|z| = 2\,\delta(z). \tag{10}$$
Also, argue that, for a well defined function $f(z)$, the replacement
$$f(z)\delta(z) = f(0)\delta(z) \tag{11}$$
is justified. Using Eq. (9), Eq. (10), and Eq. (11), verify (by substituting the solution into the differential equation) that
$$g(z) = \frac{1}{2k}\, e^{-k|z|} \tag{12}$$
is a particular solution of the differential equation
$$\left(-\frac{d^2}{dz^2} + k^2 \right) g(z) = \delta(z). \tag{13}$$

4. (**20 points.**) Verify the identity
$$\phi \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} \psi) - \psi \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} \phi) = \boldsymbol{\nabla} \cdot [\lambda(\phi \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \phi)], \tag{14}$$
which is a slight generalization of what is known as Green's second identity. Here ϕ, ψ, and λ, are position dependent functions.

5. (**20 points.**) The expression for the electric potential due to a point charge placed in front of a perfectly conducting semi-infinite slab, described by
$$\frac{\varepsilon(z)}{\varepsilon_0} = \begin{cases} \infty, & z < 0, \\ 1, & 0 < z, \end{cases} \tag{15}$$

is given in terms of the reduced Green function that satisfies the differential equation $(0 < \{z, z'\})$

$$-\left[\frac{\partial^2}{\partial z^2} - k^2\right] \varepsilon_0 g(z, z') = \delta(z - z') \tag{16}$$

with boundary conditions requiring the reduced Green's function to vanish at $z = 0$ and at $z \to \infty$.

(a) Construct the reduced Green function in the form

$$\varepsilon_0 g(z, z') = \begin{cases} Ae^{kz} + Be^{-kz}, & 0 < z < z', \\ Ce^{kz} + De^{-kz}, & 0 < z' < z, \end{cases} \tag{17}$$

and solve for the four coefficients, A, B, C, D, using the conditions

$$\varepsilon_0 g(0, z') = 0, \tag{18a}$$

$$\varepsilon_0 g(\infty, z') = 0, \tag{18b}$$

$$\varepsilon_0 g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0, \tag{18c}$$

$$\partial_z \varepsilon_0 g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1. \tag{18d}$$

(b) Express the solution in the form

$$\varepsilon_0 g(z, z') = \frac{1}{2k} e^{-k|z-z'|} - \frac{1}{2k} e^{-k|z|} e^{-k|z'|}. \tag{19}$$

6. (**20 points.**) The expression for the electric potential due to a point charge placed in between two parallel grounded perfectly conducting semi-infinite slabs, described by

$$\frac{\varepsilon(z)}{\varepsilon_0} = \begin{cases} \infty, & z < 0, \\ 1, & 0 < z < a, \\ \infty, & a < z, \end{cases} \tag{20}$$

is given in terms of the reduced Green function that satisfies the differential equation $(0 < \{z, z'\} < a)$

$$\left[-\frac{\partial^2}{\partial z^2} + k^2\right] \varepsilon_0 g(z, z') = \delta(z - z') \tag{21}$$

with boundary conditions requiring the reduced Green's function to vanish at $z = 0$ and $z = a$.

(a) Construct the reduced Green's function in the form

$$\varepsilon_0 g(z, z') = \begin{cases} A \sinh kz + B \cosh kz, & 0 < z < z' < a, \\ C \sinh kz + D \cosh kz, & 0 < z' < z < a, \end{cases} \tag{22}$$

and solve for the four coefficients, A, B, C, D, using the conditions

$$\varepsilon_0 g(0, z') = 0, \tag{23a}$$
$$\varepsilon_0 g(a, z') = 0, \tag{23b}$$
$$\varepsilon_0 g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0, \tag{23c}$$
$$\partial_z \varepsilon_0 g(z, z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1. \tag{23d}$$

(b) After using conditions in Eqs. (23a) and (23b) show that the reduced Green's function can be expressed in the form

$$\varepsilon_0 g(z, z') = \begin{cases} A \sinh kz, & 0 < z < z' < a, \\ C' \sinh k(a - z), & 0 < z' < z < a, \end{cases} \tag{24}$$

where $C' = -C/\cosh ka$. Then, use Eqs. (23c) and (23d) to show that

$$\varepsilon_0 g(z, z') = \begin{cases} \dfrac{\sinh kz \sinh k(a - z')}{k \sinh ka}, & 0 < z < z' < a, \\ \dfrac{\sinh kz' \sinh k(a - z)}{k \sinh ka}, & 0 < z' < z < a. \end{cases} \tag{25}$$

(c) Take the limit $ka \to \infty$ in your solution above, (which corresponds to moving the slab at $z = a$ to infinity,) to obtain the reduced Green's function for a single perfectly conducting slab,

$$\lim_{ka \to \infty} \varepsilon_0 g(z, z') = \frac{1}{2k} e^{-k|z-z'|} - \frac{1}{2k} e^{-k|z|} e^{-k|z'|}. \tag{26}$$

This should serve as a check for your solution to the reduced Green's function. Hint: The hyperbolic functions here are defined as

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \text{and} \qquad \cosh x = \frac{1}{2}(e^x + e^{-x}). \tag{27}$$