Homework No. 10 (Fall 2024)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Friday, 2024 Nov 22, 4.30pm

0. Problems 4, 5, 6, 7, 8, 9, and 10, are for submission.

1. (**Recurrence relation.**) The Legendre polynomials $P_l(x)$ of degree l are defined, or generated, by expanding the electric (or gravitational) potential of a point charge,

$$\frac{\alpha}{|\mathbf{r}-\mathbf{r}'|} = \frac{\alpha}{r_>} \frac{1}{\sqrt{1 + \left(\frac{r_<}{r_>}\right)^2 - 2\left(\frac{r_<}{r_>}\right)\cos\gamma}} = \frac{\alpha}{r_>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>}\right)^l P_l(\cos\gamma), \tag{1}$$

where

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'), \tag{2}$$

and

$$r_< = \text{Minimum}(r, r'), \tag{3a}$$

$$r_> = \text{Maximum}(r, r'). \tag{3b}$$

Thus, in terms of variables

$$t = \frac{r_<}{r_>}, \qquad 0 \le t < \infty, \tag{4}$$

and

$$x = \cos\gamma, \qquad -1 \le x < 1, \tag{5}$$

we can define the generating function for the Legendre polynomials as

$$g(t, x) = \frac{1}{\sqrt{1 + t^2 - 2xt}} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{6}$$

Setting $t = 0$ in the above relation we immediately learn that

$$P_0(x) = 1. \tag{7}$$

Legenendre polynomials of higher degrees can be derived by Taylor expansion of the generating function. However, for large degrees it is more efficient to derive a recurrence relation. To derive the recurrence relation for Legendre polynomials we begin by differentiating the generating function with respect to t to obtain

$$\frac{\partial g}{\partial t} = \frac{(x - t)}{(1 + t^2 - 2xt)^{\frac{3}{2}}} = \sum_{l=1}^{\infty} l\, t^{l-1} P_l(x). \tag{8}$$

Inquire why the sum on the right hand side now starts from $l = 1$. The second equality can be rewritten in the form

$$\frac{(x-t)}{\sqrt{1+t^2-2xt}} = (1+t^2-2xt)\sum_{l=1}^{\infty} l\, t^{l-1} P_l(x), \tag{9}$$

and implies

$$(x-t)\sum_{l=0}^{\infty} t^l P_l(x) = (1+t^2-2xt)\sum_{l=1}^{\infty} l\, t^{l-1} P_l(x). \tag{10}$$

Express this in the form

$$t^0\Big[xP_0(x) - P_1(x)\Big] \ +t^1\Big[3xP_1(x) - P_0(x) - 2P_2(x)\Big] \\ +\sum_{l=2}^{\infty} t^l\Big[(2l+1)x\,P_l(x) - l\,P_{l-1}(x) - (l+1)\,P_{l+1}(x)\Big] = 0. \tag{11}$$

Thus, using the completeness property of Taylor expansion, that is, equating the coefficients of powers of t in the expansion, we have, for t^0 and t^1,

$$P_1(x) = xP_0(x), \tag{12a}$$
$$2P_2(x) = 3xP_1(x) - P_0(x), \tag{12b}$$

and matching powers of t^l for $l \geq 2$ we obtain the recurrence relation for Legendre polynomials as

$$(l+1)\,P_{l+1}(x) = (2l+1)x\,P_l(x) - l\,P_{l-1}(x), \qquad l = 0,1,2,3,\ldots. \tag{13}$$

Note that the recurrence relations in Eq. (16), for $l = 0$ and $l = 1$, reproduces Eqs. (12). The recurrence relations in Eq. (16) can be reexpressed in the form

$$l\,P_l(x) = (2l-1)x\,P_{l-1}(x) - (l-1)\,P_{l-2}(x), \qquad l = 1,2,3,\ldots. \tag{14}$$

Thus, Eq. (14) generates Legendre polynomials of all degrees starting from $P_0(x) = 1$, which was obtained in Eq. (7).

2. (**Differential equation.**) The generating function for the Legendre polynomials $P_l(x)$ of degree l is

$$g(t,x) = \frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{15}$$

   (a) Starting from the generating function and differentiating with respect to t we derived the recurrence relation for Legendre polynomials in Eq. (16),

$$(l+1)\,P_{l+1}(x) = (2l+1)x\,P_l(x) - l\,P_{l-1}(x), \qquad l = 0,1,2,\ldots, \tag{16}$$

in terms of

$$P_0(x) = 1 = g(0,x). \tag{17}$$

Differentiating the recurrence relation with respect to x show that

$$(2l+1)\,P_l + (2l+1)x\,P'_l = l\,P'_{l-1} + (l+1)\,P'_{l+1}, \qquad l = 0,1,2,\ldots, \tag{18}$$

where we supressed the dependence in x and prime in the superscript of $P'_l(x)$ denotes derivative with respect to the argument x.

(b) Differentiating the generating function with respect to x show that

$$\frac{\partial g}{\partial x} = \frac{t}{(1+t^2-2xt)^{\frac{3}{2}}} = \sum_{l=0}^{\infty} t^l P'_l(x). \tag{19}$$

Show that the second equality can be rewritten in the form

$$\frac{t}{\sqrt{1+t^2-2xt}} = (1+t^2-2xt)\sum_{l=0}^{\infty} t^l P'_l(x), \tag{20}$$

and implies

$$t\sum_{l=0}^{\infty} t^l P_l(x) = (1+t^2-2xt)\sum_{l=0}^{\infty} t^l P'_l(x). \tag{21}$$

Express this in the form

$$\begin{aligned} t^0\Big[P'_0(x)\Big] \quad &+t^1\Big[P'_1(x) - 2xP'_0(x) - P_0(x)\Big] \\ &+\sum_{l=2}^{\infty} t^l\Big[P'_l(x) + P'_{l-2}(x) - 2x\,P'_{l-1}(x) - P_{l-1}(x)\Big] = 0. \end{aligned} \tag{22}$$

Then, using the completeness property of Taylor expansion, that is, equating the coefficients of powers of t in the expansion, show that, for t^0 and t^1,

$$P'_0(x) = 0, \tag{23a}$$

$$P'_1(x) = P_0(x) = 1, \tag{23b}$$

and matching powers of t^l for $l \geq 2$ derive a recurrence relation for the derivative of Legendre polynomials as

$$2x\,P'_{l-1} + P_{l-1} = P'_l + P'_{l-2}, \qquad l = 2,3,\ldots. \tag{24}$$

Here, we shall find it convenient to use the above recurrence relations in the form

$$2x\,P'_l + P_l = P'_{l+1} + P'_{l-1}, \qquad l = 1,2,3,\ldots, \tag{25}$$

which is obtained by setting $l \to l+1$.

(c) Equations (18) and (25) are linear set of equations for P'_{l-1} and P'_{l+1} in terms of P_l and P'_l. Solve them to find

$$P'_{l+1} = xP'_l + (l+1)\,P_l, \qquad l = 0,1,2,\ldots, \tag{26a}$$

$$P'_{l-1} = x\,P'_l - l\,P_l. \qquad l = 1,2,3,\ldots. \tag{26b}$$

(d) Using $l \to l-1$ in Eq. (26a) show that

$$P'_l = x\,P'_{l-1} + l\,P_{l-1}. \tag{27}$$

Then, substitute Eq. (26b) to obtain

$$(1-x^2)\,P'_l = l\,P_{l-1} - xl\,P_l. \tag{28}$$

Differentiate the above equation and substitute Eq. (26b) again to derive the differential equation for Legendre polynomials as

$$\left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x} + l(l+1)\right]P_l(x) = 0. \tag{29}$$

Substitute $x = \cos\theta$ to rewrite the differential equation in the form

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + l(l+1)\right]P_l(\cos\theta) = 0. \tag{30}$$

3. (**Rodrigues formula for Legendre polynomials**.)
The generating function for the Legendre polynomials $P_l(x)$ of degree l is

$$g(t,x) = \frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{31}$$

(a) Using binomial expansion show that

$$\frac{1}{\sqrt{1-y}} = \sum_{m=0}^{\infty} y^m \frac{(2m)!}{\left[m!\,2^m\right]^2} \tag{32}$$

and

$$(2xt-t^2)^m = \sum_{n=0}^{\infty} \frac{m!}{n!(m-n)!}(2xt)^{m-n} t^{2n} (-1)^n. \tag{33}$$

Thus, show that

$$\frac{1}{\sqrt{1+t^2-2xt}} = \sum_{m=0}^{\infty}\sum_{n=0}^{m} t^{m+n} \frac{(2m)!}{m!n!(m-n)!2^{m+n}} x^{m-n}(-1)^n. \tag{34}$$

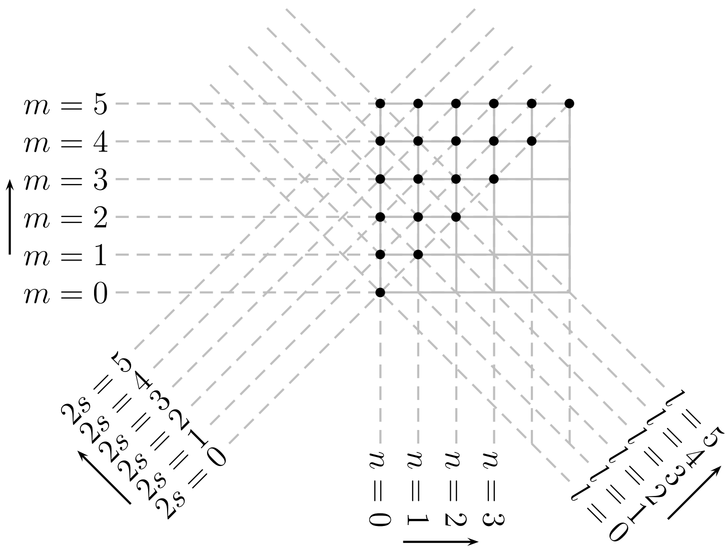

Figure 1: Double summation.

(b) In Figure 1 we illustrate how we change the double sum in m and n to variables l and s. This is achieved using the substitutions

$$m + n = l, \tag{35a}$$

$$m - n = 2s, \tag{35b}$$

which corresponds to

$$2m = l + 2s, \qquad m = \frac{l}{2} + s, \quad \text{and} \quad n = \frac{l}{2} - s. \tag{36}$$

The counting on the variable s, for given l, follows the pattern,

$$l \text{ even}: \quad 2s = 0, 2, 4, \ldots, l, \tag{37a}$$

$$l \text{ odd}: \quad 2s = 1, 3, 5, \ldots, l. \tag{37b}$$

Show that in terms of l and s the double summation can be expressed as

$$\frac{1}{\sqrt{1 + t^2 - 2xt}} = \sum_{l=0}^{\infty} \sum_{s} t^l \frac{(l + 2s)!}{\left(\frac{l}{2} + s\right)! \left(\frac{l}{2} - s\right)! (2s)! 2^l} x^{2s} (-1)^{\frac{l}{2} - s}, \tag{38}$$

where the limits on the sum in s are dictated by Eqs. (37) depending on l being even or odd. Thus, read out the polynomial expression for Legendre polynomials of degree l to be

$$P_l(x) = \sum_{s} \frac{(l + 2s)!}{\left(\frac{l}{2} + s\right)! \left(\frac{l}{2} - s\right)! (2s)! 2^l} x^{2s} (-1)^{\frac{l}{2} - s}, \tag{39}$$

where the summation on s depends on whether l is even or odd.

(c) Show that

$$\left(\frac{d}{dx}\right)^l x^{l+2s} = \frac{(l+2s)!}{(2s)!} x^{2s}. \tag{40}$$

Thus, show that

$$P_l(x) = \frac{1}{l!\,2^l} \left(\frac{d}{dx}\right)^l \sum_s \frac{l!}{\left(\frac{l}{2}+s\right)!\left(\frac{l}{2}-s\right)!} x^{l+2s} (-1)^{\frac{l}{2}-s}. \tag{41}$$

(d) For even l the summation in s runs from $s=0$ to $s=l/2$, Thus, writing $l+2s = 2[l - \left(\frac{l}{2}-s\right)]$, show that

$$P_l(x) = \frac{1}{l!\,2^l} \left(\frac{d}{dx}\right)^l \sum_{s=0}^{\frac{l}{2}} \frac{l!}{\left(\frac{l}{2}+s\right)!\left(\frac{l}{2}-s\right)!} (x^2)^{l-\left(\frac{l}{2}-s\right)} (-1)^{\left(\frac{l}{2}-s\right)}. \tag{42}$$

Then, substituting

$$\frac{l}{2} - s = n, \tag{43}$$

show that

$$P_l(x) = \frac{1}{l!\,2^l} \left(\frac{d}{dx}\right)^l \sum_{n=0}^{\frac{l}{2}} \frac{l!}{(l-n)!n!} (x^2)^{l-n} (-1)^n. \tag{44}$$

Note that the summation on n runs from $n=0$ to $n=l/2$. If we were to extend this sum to $n=l$ verify that the additional terms will have powers in x less than l. Since the terms in the sum are acted upon by l derivatives with respect to x these additional terms will not contribute. Thus, show that

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{l!\,2^l}. \tag{45}$$

Similarly, for odd l the summation is s runs as

$$2s = 1, 3, 5, \ldots, l, \tag{46}$$

or

$$\frac{2s-1}{2} = 0, 1, 2, \ldots, \frac{l-1}{2}. \tag{47}$$

Thus, substituting

$$s' = \frac{2s-1}{2} = s - \frac{1}{2}, \tag{48}$$

show that

$$P_l(x) = \frac{1}{l!\,2^l} \left(\frac{d}{dx}\right)^l \sum_{s=0}^{\frac{l-1}{2}} \frac{l!}{\left(\frac{l+1}{2}+s\right)!\left(\frac{l-1}{2}-s\right)!} x^{l+1+2s} (-1)^{\left(\frac{l-1}{2}-s\right)}. \tag{49}$$

Substituting

$$\frac{l-1}{2} - s = n \tag{50}$$

and writing

$$\frac{l+1}{2} + s = l - \left(\frac{l-1}{2} - s\right) \tag{51}$$

show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{n=0}^{\frac{l-1}{2}} \frac{l!}{(l-n)!n!}(x^2)^{l-n}(-1)^n. \tag{52}$$

Again, like in the case of even l we can extend the sum on n beyond $n = (l-1)/2$, because they do not survive under the action of l derivatives with respect to x. Thus, again, we have

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{l!\,2^l}, \tag{53}$$

which is exactly the form obtained for even l. The expression in Eq. (53) is the Rodrigues formula for generating the Legendre polynomials of degree l.

4. (**20 points.**) Using Mathematica (or another graphing tool) plot the Legendre polynomials $P_l(x)$ for $l = 0, 1, 2, 3, 4$ on the same plot. Note that $-1 \leq x \leq 1$. Based on the pattern you see what can you conclude about the number of roots for $P_l(x)$. In Mathematica these plots are generated using the following commands:
`Plot[{LegendreP[0,x], LegendreP[1,x], LegendreP[2,x], LegendreP[3,x], LegendreP[4,x] },{x,-1,1}]`
Compare your plots with those in Wikipedia article on 'Legendre Polynomials'. While there read the Wikipedia article on Adrien-Marie Legendre and the associated 'Portrait Debacle'.

5. (**20 points.**) Legendre polynomials are conveniently generated using the relation

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}, \tag{54}$$

where $-1 \leq x \leq 1$. Evaluate Legendre polynomials of degree $l = 0, 1, 2, 3, 4$ in this manner.

6. (**20 points.**) Legendre polynomials $P_l(x)$ satisfy the relation

$$\int_{-1}^{1} dx\, P_l(x) = 0 \quad \text{for} \quad l \geq 1. \tag{55}$$

Verify this explicitly for $l = 0, 1, 2, 3, 4$.

7. (**20 points.**) Legendre polynomials satisfy the differential equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + l(l+1)\right] P_l(\cos\theta) = 0. \tag{56}$$

Verify this explicitly for $l = 0, 1, 2, 3, 4$.

8. (**20 points.**) Legendre polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}. \tag{57}$$

Verify this explicitly for $l = 0, 1, 2$ and $l' = 0, 1, 2$. The orthogonality relation is also expressed as

$$\int_0^\pi \sin\theta d\theta\, P_l(\cos\theta) P_{l'}(\cos\theta) = \frac{2}{2l+1}\delta_{ll'}. \tag{58}$$

9. (**20 points.**) Express the function

$$\sigma(\theta) = \cos^2\theta \tag{59}$$

in terms of Legendre polynomials.
Solution:

$$\sigma(\theta) = \frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_0(\cos\theta). \tag{60}$$

10. (**20 points.**) Recollect Legendre polynomials of order l

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}. \tag{61}$$

In particular

$$P_0(x) = 1, \tag{62a}$$
$$P_1(x) = x, \tag{62b}$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}. \tag{62c}$$

Consider a charged spherical shell of radius a consisting of a charge distribution in the polar angle alone,

$$\rho(\mathbf{r}') = \sigma(\theta')\,\delta(r' - a). \tag{63}$$

The electric potential *on the z-axis*, $\theta = 0$ and $\phi = 0$, is then given by

$$\begin{aligned}\phi(r, 0, 0) &= \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\ &= \frac{2\pi a^2}{4\pi\varepsilon_0}\int_0^\pi \sin\theta' d\theta' \frac{\sigma(\theta')}{\sqrt{r^2 + a^2 - 2ar\cos\theta'}},\end{aligned} \tag{64}$$

after evaluating the r' and ϕ' integral.

(a) Consider a uniform charge distribution on the shell,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_0(\cos\theta). \tag{65}$$

Evaluate the integral in Eq. (64) to show that

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_>}, \tag{66}$$

where $r_< = \text{Min}(a,r)$ and $r_> = \text{Max}(a,r)$.

(b) Next, consider a (pure dipole, 2×1-pole,) charge distribution of the form,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_1(\cos\theta). \tag{67}$$

Evaluate the integral in Eq. (64) to show that

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{3}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right). \tag{68}$$

(c) Next, consider a (pure quadrapole, 2×2-pole,) charge distribution of the form,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_2(\cos\theta). \tag{69}$$

Evaluate the integral in Eq. (64) to show that

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{5}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^2. \tag{70}$$

(d) For a (pure $2l$-pole) charge distribution

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_l(\cos\theta) \tag{71}$$

the integral in Eq. (64) leads to

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{(2l+1)}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l. \tag{72}$$