# Homework No. 11 (Fall 2024)

## PHYS 500A: MATHEMATICAL METHODS

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Due date: Friday, 2024 Dec 6, 4.30pm

1. (**20 points.**) Inversion is a transformation that maps a point $\mathbf{r}$ inside (outside) a sphere of radius $a$ to a point

$$\mathbf{r}_a = \frac{a^2}{r^2}\mathbf{r} \tag{1}$$

outside (inside) the sphere. Given that the function $\phi(\mathbf{r})$ satisfies the Laplacian,

$$\nabla^2\phi(\mathbf{r}) = 0, \tag{2}$$

show that

$$\frac{a}{r}\,\phi\left(\frac{a^2}{r^2}\mathbf{r}\right) \tag{3}$$

also satisfies the Laplacian for $r \neq 0$. That is,

$$\nabla^2\left[\frac{a}{r}\,\phi\left(\frac{a^2}{r^2}\mathbf{r}\right)\right] = 0. \tag{4}$$

To this end, using Eq. (1) evaluate $\mathbf{r}_a \cdot \mathbf{r}_a$ and thus derive

$$r_a r = a^2. \tag{5}$$

Then, show that

$$\frac{a}{r}\,\phi\left(\frac{a^2}{r^2}\mathbf{r}\right) = \frac{r_a}{a}\,\phi(\mathbf{r}_a). \tag{6}$$

To express the gradient in terms of the inverted variable $\mathbf{r}_a$ write

$$\boldsymbol{\nabla} = \frac{\partial}{\partial\mathbf{r}} = \frac{\partial\mathbf{r}_a}{\partial\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{r}_a} = (\boldsymbol{\nabla}\,\mathbf{r}_a)\cdot\boldsymbol{\nabla}_a. \tag{7}$$

Show that

$$(\boldsymbol{\nabla}\,\mathbf{r}_a) = \frac{1}{a^2}(\mathbf{1}\,r_a^2 - 2\,\mathbf{r}_a\,\mathbf{r}_a). \tag{8}$$

Thus, show that

$$\boldsymbol{\nabla} = \frac{1}{a^2}(\mathbf{1}\,r_a^2 - 2\,\mathbf{r}_a\,\mathbf{r}_a)\cdot\boldsymbol{\nabla}_a \tag{9}$$

and

$$\nabla^2 = \frac{1}{a^4}\left[(\mathbf{1}\,r_a^2 - 2\,\mathbf{r}_a\,\mathbf{r}_a)\cdot\boldsymbol{\nabla}_a\right]\cdot\left[(\mathbf{1}\,r_a^2 - 2\,\mathbf{r}_a\,\mathbf{r}_a)\cdot\boldsymbol{\nabla}_a\right]. \tag{10}$$

Expand the operations and simplify to derive

$$a^4\nabla^2 = r_a^4\nabla_a^2 - 2r_a^2\mathbf{r}_a \cdot \boldsymbol{\nabla}_a. \tag{11}$$

To prove the statement in Eq. (4) show that

$$\nabla^2 \left[ \frac{a}{r} \phi \left( \frac{a^2}{r^2}\mathbf{r} \right) \right] = \frac{r_a^5}{a^5}\nabla_a^2 \phi(\mathbf{r}_a) = 0. \tag{12}$$

2. (**20 points.**) The fundamental solution to Laplace's equation is the electric potential due to a point charge,

$$\frac{q}{4\pi\varepsilon_0}\frac{1}{r}. \tag{13}$$

Dropping $q/(4\pi\varepsilon_0)$ we have

$$\nabla^2 \frac{1}{r} = 0, \quad r \neq 0. \tag{14}$$

In terms of this solution, we can generate a large number of others. For example, for constant vectors $\mathbf{s}_1$,

$$\nabla^2 (\mathbf{s}_1 \cdot \boldsymbol{\nabla})\frac{1}{r} = 0. \tag{15}$$

Solid harmonics of degree $-(l+1)$ are defined as

$$V_l(\mathbf{r}) = \frac{1}{l!}(-\mathbf{s}_1 \cdot \boldsymbol{\nabla})(-\mathbf{s}_2 \cdot \boldsymbol{\nabla}) \ldots (-\mathbf{s}_l \cdot \boldsymbol{\nabla})\frac{1}{r} \tag{16}$$

for $l = 1, 2, \ldots,$ with

$$V_0(\mathbf{r}) = \frac{1}{r} \tag{17}$$

for $l = 0$. Solid harmonics satisfy the Laplace equation. It is of interest to see the form of the solid harmonics after the gradient operations have been evaluated.

(a) Define

$$\mu_i = (\mathbf{s}_i \cdot \hat{\mathbf{r}}), \qquad \tilde{\mu}_i = (\mathbf{s}_i \cdot \mathbf{r}), \tag{18a}$$

$$\lambda_{ij} = (\mathbf{s}_i \cdot \mathbf{s}_j), \qquad \tilde{\lambda}_{ij} = (\mathbf{s}_i \cdot \mathbf{s}_j)r^2. \tag{18b}$$

Show that

$$(-\mathbf{s}_i \cdot \boldsymbol{\nabla})\tilde{\mu}_j = -\frac{\tilde{\lambda}_{ij}}{r^2}, \tag{19a}$$

$$(-\mathbf{s}_i \cdot \boldsymbol{\nabla})\frac{1}{r^m} = \frac{m}{r^{m+2}}\,\tilde{\mu}_i, \tag{19b}$$

$$(-\mathbf{s}_k \cdot \boldsymbol{\nabla})\tilde{\lambda}_{ij} = -\frac{2\tilde{\mu}_k\tilde{\lambda}_{ij}}{r^2}. \tag{19c}$$

(b) Show that

$$
\begin{aligned}
V_1 &= \frac{1}{1!}\frac{1}{r^3}\Big[\tilde{\mu}_1\Big], &(20a)\\
V_2 &= \frac{1}{2!}\frac{1}{r^5}\Big[3\tilde{\mu}_1\tilde{\mu}_2 - \tilde{\lambda}_{12}\Big], &(20b)\\
V_3 &= \frac{1}{3!}\frac{1}{r^7}\Big[15\tilde{\mu}_1\tilde{\mu}_2\tilde{\mu}_3 - 3\tilde{\mu}_1\tilde{\lambda}_{23} - 3\tilde{\mu}_2\tilde{\lambda}_{31} - 3\tilde{\mu}_3\tilde{\lambda}_{12}\Big], &(20c)\\
V_4 &= \frac{1}{4!}\frac{1}{r^9}\Big[105\tilde{\mu}_1\tilde{\mu}_2\tilde{\mu}_3\tilde{\mu}_4 - 15\tilde{\mu}_1\tilde{\mu}_2\tilde{\lambda}_{34} - 15\tilde{\mu}_1\tilde{\mu}_3\tilde{\lambda}_{24} - 15\tilde{\mu}_1\tilde{\mu}_4\tilde{\lambda}_{23} - 15\tilde{\mu}_2\tilde{\mu}_3\tilde{\lambda}_{14}\\
&\qquad -15\tilde{\mu}_2\tilde{\mu}_4\tilde{\lambda}_{13} - 15\tilde{\mu}_3\tilde{\mu}_4\tilde{\lambda}_{12} + 3\tilde{\lambda}_{12}\tilde{\lambda}_{34} + 3\tilde{\lambda}_{13}\tilde{\lambda}_{24} + 3\tilde{\lambda}_{34}\tilde{\lambda}_{12}\Big]. &(20d)
\end{aligned}
$$

For bringing compactness we introduce the notation

$$\mu^{l-2m}\lambda^m = \mu_1\mu_2\ldots\mu_{l-2m}\lambda_{..}\lambda_{..}\ldots + \text{combinations} \tag{21}$$

in terms of which we find

$$V_l = \frac{1}{r^{l+1}}\sum_{m=0}^{l-1}\frac{(2[l-m])!}{2^{l-m}l!(l-m)!}(-1)^m\mu^{l-2m}\lambda^m. \tag{22}$$

(c) Surface (or spherical) harmonics $Y_l$ of degree $l$ are defined using the relation

$$V_l = \frac{Y_l}{r^{l+1}}. \tag{23}$$

Show that

$$Y_l = \sum_{m=0}^{l-1}\frac{(2[l-m])!}{2^{l-m}l!(l-m)!}(-1)^m\mu^{l-2m}\lambda^m. \tag{24}$$

(d) Solid harmonics $H_l$ of degree $l$ are defined using the relation

$$V_l = \frac{H_l}{r^{2l+1}}. \tag{25}$$

Show that

$$H_l = \sum_{m=0}^{l-1}\frac{(2[l-m])!}{2^{l-m}l!(l-m)!}(-1)^m\tilde{\mu}^{l-2m}\tilde{\lambda}^m. \tag{26}$$

(e) Zonal harmonics $P_l$ of order $l$ are defined to be surface harmonics of degree $l$ with the special choice

$$\mathbf{s}_1 = \mathbf{s}_2 = \cdots = \mathbf{s}_l = \mathbf{s}. \tag{27}$$

Then, $\lambda_{ij} = 1$ and all $\mu_i$'s are identical, say $\mu$. Show that

$$P_l = \sum_{m=0}^{l-1}\frac{(2[l-m])!}{2^l m!(l-m)!(l-2m)!}(-1)^m\mu^{l-2m}. \tag{28}$$

Recognize the zonal harmonics as the Legendre polynomials.

3. (**20 points.**) Using the definition of spherical harmonics

$$Y_{lm}(\theta,\phi) = e^{im\phi}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\frac{1}{(\sin\theta)^m}\left(\frac{d}{d\cos\theta}\right)^{l-m}\frac{(\cos^2\theta-1)^l}{2^l l!}, \tag{29}$$

evaluate the explicit expressions for $Y_{00}$, $Y_{11}$, $Y_{10}$, $Y_{1,-1}$, $Y_{22}$, $Y_{21}$, $Y_{20}$, $Y_{2,-1}$, and $Y_{2,-2}$.

4. (**40 points.**) Generate 3D plots of surface spherical harmonics $Y_{lm}(\theta,\phi)$ as a function of $\theta$ and $\phi$. In particular,

   (a) Plot $\mathrm{Re}\big[Y_{73}(\theta,\phi)\big]$.

   (b) Plot $\mathrm{Im}\big[Y_{73}(\theta,\phi)\big]$.

   (c) Plot $\mathrm{Abs}\big[Y_{73}(\theta,\phi)\big]$.

   (d) Plot your favourite spherical harmonic, that is, choose a $l$ and $m$, and Re or Im or Abs.

   Hint: In Mathematica these plots are generated using the following commands:
   ```
   SphericalPlot3D[Re[SphericalHarmonicY[l,m,θ,ϕ]],{θ,0,Pi},{ ϕ,0,2 Pi}]
   SphericalPlot3D[Im[SphericalHarmonicY[l,m,θ,ϕ]],{θ,0,Pi},{ ϕ,0,2 Pi}]
   SphericalPlot3D[Abs[SphericalHarmonicY[l,m,θ,ϕ]],{θ,0,Pi},{ ϕ,0,2 Pi}]
   ```
   Refer diagrams in Wikipedia article on 'spherical harmonics' to see some visual representations of these functions.