Final Exam (2025 Spring)

PHYS 510: CLASSICAL MECHANICS

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Date: 2025 May 8

1. (**20 points.**) The Kepler problem is described by the Lagrangian

$$L(r,\phi,\dot{r},\dot{\phi}) = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + \frac{\alpha}{r}, \tag{1}$$

where the first term is the contribution to the kinetic energy of the particle with reduced mass μ due to radial velocity $\dot{r}$, the second term is the contribution to the kinetic energy due to it's tangential velocity $\dot{\phi}$, and the third term is the negative of the gravitational potential energy between masses m_1 and m_2. Here $\alpha = G\mu M$. (Show that $\mu M = m_1 m_2$.) Show that the canonical momentum in the radial direction and the associated force are

$$p_r = \frac{\partial L}{\partial \dot{r}} = \mu\dot{r}, \tag{2a}$$

$$F_r = \frac{\partial L}{\partial r} = \mu r\dot{\phi}^2 - \frac{\alpha}{r^2}, \tag{2b}$$

respectively. Thus, derive the equation for the radial motion to be

$$\frac{d}{dt}\mu\dot{r} = \mu r\dot{\phi}^2 - \frac{\alpha}{r^2}, \tag{3}$$

where the second term on the right is the gravitational force of attraction and the first term on the right is the centrifugal force due to the continuous change in the direction of tangential velocity. Show that the canonical momentum in the tangential direction, the angular momentum, and the associated canonical force, the torque, are

$$L_z = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2\dot{\phi}, \tag{4a}$$

$$F_\phi = \frac{\partial L}{\partial \phi} = 0, \tag{4b}$$

so that the angular momentum is a constant of motion,

$$\frac{d}{dt}L_z = 0. \tag{5}$$

(a) Using the conservation of angular momentum in Eq. (4a) to replace $\dot{\phi}$ in the equation of motion in Eq. (3) derive

$$\frac{d}{dt}\mu\dot{r} = \frac{L_z^2}{\mu r^3} - \frac{\alpha}{r^2}, \tag{6}$$

such that we can write

$$\frac{d}{dt}\mu\dot{r} = \frac{\partial}{\partial r}\left(-\frac{L_z^2}{2\mu r^2} + \frac{\alpha}{r}\right). \tag{7}$$

Multiplying by $\dot{r}$ on both sides gives

$$\dot{r}\frac{d}{dt}\mu\dot{r} = \frac{dr}{dt}\frac{\partial}{\partial r}\left(-\frac{L_z^2}{2\mu r^2} + \frac{\alpha}{r}\right) \tag{8}$$

which can be written in the form

$$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2 + \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r}\right) = 0 \tag{9}$$

and is interpreted as the statement of conservation of energy.

(b) Find the error in the following steps. Using the conservation of angular momentum in Eq. (4a) to replace $\dot{\phi}$ in the Lagrangian in Eq. (1) derive

$$L(r,\dot{r}) = \frac{1}{2}\mu\dot{r}^2 + \frac{L_z^2}{2\mu r^2} + \frac{\alpha}{r} \tag{10}$$

and derive the equation of motion

$$\frac{d}{dt}\mu\dot{r} = \frac{\partial}{\partial r}\left(\frac{L_z^2}{2\mu r^2} + \frac{\alpha}{r}\right). \tag{11}$$

Thus, derive the statement of conservation of energy as

$$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2 - \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r}\right) = 0 \tag{12}$$

with a wrong sign.

2. (**20 points.**) The effective potential energy for the Kepler problem is

$$U_{\text{eff}}(r) = \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r}, \tag{13}$$

where the first term is the energy associated with the centrifugal force and the second term is the gravitational potential energy. Show that the equilibrium point for the above potential energy function is

$$r_0 = \frac{L_z^2}{\mu\alpha} \tag{14}$$

and the corresponding minimum energy is

$$U_{\text{eff}}(r_0) = -\frac{\alpha}{2r_0}. \tag{15}$$

For total energy $E < 0$ show that the potential energy function have two turning points,

$$r_{\rm min} = \frac{r_0}{1+e} \tag{16}$$

and

$$r_{\rm max} = \frac{r_0}{1-e}, \tag{17}$$

where the eccentricity e is given by

$$e = \sqrt{1 - \frac{E}{U_{\rm eff}(r_0)}}. \tag{18}$$

Next, consider a perturbation to the effective potential energy,

$$U'_{\rm eff}(r) = \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r} + \frac{\beta_3}{r^3}, \tag{19}$$

such that

$$\kappa = \frac{\beta_3/r_0^3}{\alpha/r_0} = \frac{\beta_3}{\alpha r_0^2} \ll 1. \tag{20}$$

To the leading order in κ, show that the shift in the equilibrium point is

$$r'_0 = r_0(1+3\kappa) \tag{21}$$

and the leading order shift in the minimum energy is

$$U'_{\rm eff}(r'_0) = U_{\rm eff}(r_0)\Big[1 - 2\kappa\Big]. \tag{22}$$

Show that the leading order shifts in the turning points are

$$r'_{\rm min} = r_{\rm min}\left[1 + \kappa\frac{(1+e)^2}{e}\right] \tag{23}$$

and

$$r'_{\rm max} = r_{\rm max}\left[1 - \kappa\frac{(1-e)^2}{e}\right]. \tag{24}$$

After the perturbation the trajectory is no more an ellipse. Nevertheless, for small perturbation we can define the leading order shift in the eccentricity using

$$e' = \frac{r'_{\rm max} - r'_{\rm min}}{r'_{\rm max} + r'_{\rm min}}. \tag{25}$$

Evaluate

$$e' = e\left[1 - \kappa\frac{(1-e^2)^2}{e^2}\right]. \tag{26}$$

Illustrate the above shifts in the plot for effective potential energy.

3. (**20 points.**) The path of a relativistic particle 1 moving along a straight line with constant (proper) acceleration g is described by the equation of a hyperbola

$$z_1(t) = \sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{27}$$

This is the motion of a particle that comes to existance at $z_1 = +\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = z_0$, and then returns back to infinity. Consider another relavistic particle 2 undergoing hyperbolic motion given by

$$z_2(t) = -\sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{28}$$

This is the motion of a particle that comes to existance at $z_2 = -\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = -z_0$, and then returns back to negative infinity. The world-line of particle 1 is the blue curve in Figure 3, and the world-line of particle 2 is the red curve in Figure 3. Using geometric (diagrammatic) arguments might be easiest to answer the following. Imagine the particles are sources of light (imagine a flash light pointing towards origin).

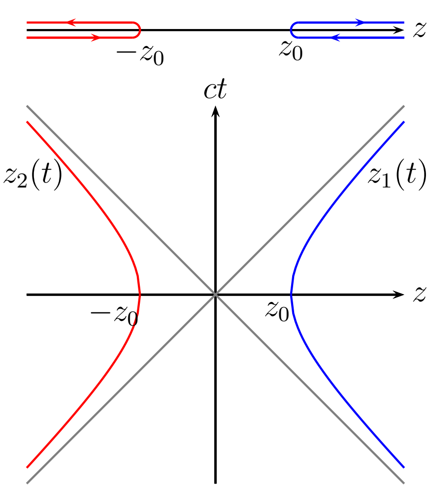

Figure 1: Problem 3

(a) At what time will the light from particle 1 first reach particle 2? Where are the particles when this happens?

(b) At what time will the light from particle 2 first reach particle 1? Where are the particles when this happens?

(c) Can the particles communicate with each other?

(d) Can the particles ever detect the presence of the other? In other words, can one particle be aware of the existence of the other? What can you deduce about the observable part of our universe from this analysis?

4. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration α is described by equation of a hyperbola

$$z^2 - c^2t^2 = z_0^2, \qquad z_0 = \frac{c^2}{\alpha}. \tag{29}$$

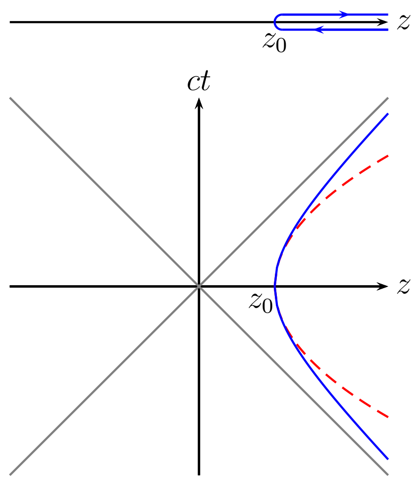

Figure 2: Problem 4

(a) This represents the world-line of a particle thrown from $z > z_0$ at $t < 0$ towards $z = z_0$ in region of constant (proper) acceleration α as described by the bold (blue) curve in the space-time diagram in Figure 4. In contrast a Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$z - z_0 = \frac{1}{2}\alpha t^2 \tag{30}$$

as described by the dashed (red) curve in the space-time diagram in Figure 4. Show that the hyperbolic curve

$$z = z_0\sqrt{1 + \frac{c^2t^2}{z_0^2}} \tag{31}$$

in regions that satisfy

$$t \ll \frac{c}{\alpha} \tag{32}$$

is approximately the parabolic curve

$$z = z_0 + \frac{1}{2}\alpha t^2 + \ldots. \tag{33}$$

(b) Recognize that the proper acceleration α does not have an upper bound.

(c) A large acceleration is achieved by taking an above turn while moving very fast. Thus, turning around while moving close to the speed of light c should achieve the highest acceleration. Show that $\alpha \to \infty$ corresponding to $z_0 \to 0$ represents this scenario. What is the equation of motion of a particle moving with infinite proper acceleration. To gain insight, plot world-lines of particles moving with $\alpha = c^2/z_0$, $\alpha = 10c^2/z_0$, and $\alpha = 100c^2/z_0$.

5. (**20 points.**) A relativisitic particle in a uniform electric field is described by the equations

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v}, \tag{34a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{34b}$$

where

$$E = mc^2\gamma, \tag{35a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{35b}$$

and

$$\mathbf{F} = q\mathbf{E}. \tag{36}$$

Let us consider the configuration with the electric field in the $\hat{\mathbf{y}}$ direction,

$$\mathbf{E} = E\,\hat{\mathbf{y}}, \tag{37}$$

and initial conditions

$$\mathbf{v}(0) = 0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{38a}$$

$$\mathbf{x}(0) = 0\,\hat{\mathbf{x}} + y_0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}. \tag{38b}$$

(a) In terms of the definition

$$\boldsymbol{\omega}_0 = \frac{1}{c}\frac{q\mathbf{E}}{m}, \tag{39}$$

show that the equations of motion are given by

$$\frac{d\gamma}{dt} = \boldsymbol{\omega}_0 \cdot \boldsymbol{\beta} \tag{40}$$

and

$$\frac{d}{dt}(\boldsymbol{\beta}\gamma) = \boldsymbol{\omega}_0. \tag{41}$$

(b) Since the particle starts from rest show that we have

$$\boldsymbol{\beta}\gamma = \boldsymbol{\omega}_0 t. \tag{42}$$

For our configuration this implies

$$\beta_x = 0, \tag{43a}$$
$$\beta_y \gamma = \omega_0 t, \tag{43b}$$
$$\beta_z = 0. \tag{43c}$$

Further, deduce

$$\beta_y = \frac{\omega_0 t}{\sqrt{1 + \omega_0^2 t^2}}. \tag{44}$$

Integrate again and use the initial condition to show that the motion is described by

$$y - y_0 = \frac{c}{\bar{\omega}_0} \left[\sqrt{1 + \bar{\omega}_0^2 t^2} - 1 \right]. \tag{45}$$

Rewrite the solution in the form

$$\left(y - y_0 + \frac{c}{\omega_0} \right)^2 - c^2 t^2 = \frac{c^2}{\omega_0^2}. \tag{46}$$

This represents a hyperbola passing through $y = y_0$ at $t = 0$. If we choose the initial position $y_0 = c/\omega_0$ we have

$$y^2 - c^2 t^2 = y_0^2. \tag{47}$$

(c) The (constant) proper acceleration associated with this motion is

$$\alpha = \omega_0 c = \frac{c^2}{y_0}. \tag{48}$$

A Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$y - y_0 = \frac{1}{2} \alpha t^2. \tag{49}$$

Show that the hyperbolic curve

$$y = y_0 \sqrt{1 + \frac{c^2 t^2}{y_0^2}} \tag{50}$$

in regions that satisfy

$$\omega_0 t \ll 1 \tag{51}$$

is approximately the parabolic curve

$$y = y_0 + \frac{1}{2} \alpha t^2 + \ldots. \tag{52}$$