Homework No. 02 (2025 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale
Due date: Tuesday, 2025 Jan 28, 4.30pm

0. (**Resource**, No submission needed.) The following classroom lecture from Spring 2024,

https://youtu.be/x05ZdUxzOKo,

serves as a good resource for functional derivative.

(a) In discrete multi-variable calculus we have a function

$$f(y^i) \tag{1}$$

dependent on variables

$$y^i, \qquad i = 1, 2, \ldots, \tag{2}$$

such that for each i we have the derivative

$$\frac{\partial f}{\partial y^i} = \lim_{\Delta y^i \to 0} \frac{f(y^j + \Delta y^j) - f(y^j)}{\Delta y^i} \tag{3}$$

evaluated in such a way that the variation in y^j is independent of a variation in y^i unless $i = j$, that is,

$$\frac{\partial y^j}{\partial y^i} = \delta_i{}^j, \tag{4}$$

where $\delta_i{}^j$ is the Kronecker delta symbol.

(b) In continuous multi-variable calculus we have a functional

$$F[y] \tag{5}$$

dependent on functions

$$y(x), \qquad x_1 < x < x_2, \tag{6}$$

such that for each x we have the derivative

$$\frac{\delta F[y]}{\delta y(x)} = \lim_{\Delta y(x) \to 0} \frac{F(y + \Delta y) - F(y)}{\Delta y(x)} \tag{7}$$

evaluated in such a way that the variation in $y(x')$ is independent of a variation in $y(x)$ unless $x = x'$, that is,

$$\frac{\delta y(x')}{\delta y(x)} = \delta(x - x'), \tag{8}$$

where $\delta(x - x')$ is the Dirac delta function.

(c) The vector form of the fundamental functional derivative is

$$\frac{\delta \mathbf{r}(s)}{\delta \mathbf{r}(s')} = \mathbf{1}\delta(s - s'). \tag{9}$$

As an illustration, we evaluate the functional derivative

$$\frac{\delta}{\delta \mathbf{r}(s')}\frac{1}{r(s)}, \tag{10}$$

where $r(s)$ is the magnitude of the vector $\mathbf{r}(s)$, as

$$\frac{\delta}{\delta \mathbf{r}(s')}\frac{1}{r(s)} = \frac{\delta}{\delta \mathbf{r}(s')}\frac{1}{\sqrt{\mathbf{r}(s)\cdot\mathbf{r}(s)}} \tag{11a}$$

$$= -\frac{1}{2}\frac{2\,\mathbf{r}(s)}{(\mathbf{r}(s)\cdot\mathbf{r}(s))^{\frac{3}{2}}}\cdot\frac{\delta \mathbf{r}(s)}{\delta \mathbf{r}(s')} \tag{11b}$$

$$= -\frac{\mathbf{r}(s)}{r(s)^3}\delta(s-s'). \tag{11c}$$

1. (**20 points.**) The principal identity of functional differentiation is

$$\frac{\delta u(x)}{\delta u(x')} = \delta(x-x'), \tag{12}$$

which states that the variation in the function u at x is independent of the variation in the function u at x' unless $x=x'$. This is a generalization of the identity in multivariable calculus

$$\frac{\partial u^j}{\partial u^i} = \delta_i{}^j, \tag{13}$$

which states that the variables u^i and u^j are independent unless $i=j$. Using the property of δ-function,

$$\int_{-\infty}^{\infty} dx\, a(x)\delta(x-x') = a(x'), \tag{14}$$

derive the following identities by repeatedly differentiating by parts.

(a)

$$\int_{-\infty}^{\infty} dx\, a(x)\frac{d}{dx}\delta(x-x') = -\frac{d}{dx'}a(x') \tag{15}$$

(b)

$$\int_{-\infty}^{\infty} dx\, a(x)\frac{d^2}{dx^2}\delta(x-x') = +\frac{d^2}{dx'^2}a(x') \tag{16}$$

(c)

$$\int_{-\infty}^{\infty} dx\, a(x)\frac{d^3}{dx^3}\delta(x-x') = -\frac{d^3}{dx'^3}a(x') \tag{17}$$

(d)

$$\int_{-\infty}^{\infty} dx\, a(x)\frac{d^n}{dx^n}\delta(x-x') = (-1)^n\frac{d^n}{dx'^n}a(x') \tag{18}$$

2. (**20 points.**) Evaluate the functional derivative

$$\frac{\delta F[u]}{\delta u(x)} \tag{19}$$

of the following functionals, assuming no variation at the end points.

(a)

$$F[u] = \int_{x_1}^{x_2} dx\, a(x)u(x) \tag{20}$$

(b)

$$F[u] = \int_{x_1}^{x_2} dx\, a(x) u(x)^2 \tag{21}$$

(c)

$$F[u] = \int_{x_1}^{x_2} dx\, \sqrt{1 + u(x)^2} \tag{22}$$

(d)

$$F[u] = \int_{x_1}^{x_2} dx\, \Big[u(x) + a(x)\Big] \Big[u(x) + b(x)\Big] \tag{23}$$

(e)

$$F[u] = \int_{x_1}^{x_2} dx\, \frac{a(x)u(x)}{\Big[1 + b(x)u(x)\Big]} \tag{24}$$

3. (**20 points.**) [Refer: Gelfand and Fomin, Calculus of Variations.] Evaluate the functional derivative

$$\frac{\delta F[y]}{\delta y(x)} \tag{25}$$

of the following functionals, assuming no variation at the end points.

(a)

$$F[y] = \int_0^1 dx\, \frac{dy}{dx} \tag{26}$$

(b)

$$F[y] = \int_{x_1}^{x_2} dx\, a(x) \frac{dy(x)}{dx} \tag{27}$$

(c)

$$F[y] = \int_0^1 dx\, y \frac{dy}{dx} \tag{28}$$

(d)

$$F[y] = \int_0^1 dx\, xy \frac{dy}{dx} \tag{29}$$

(e)

$$F[y] = \int_a^b \frac{dx}{x^3} \left(\frac{dy}{dx}\right)^2 \tag{30}$$

4. (**20 points.**) Evaluate the functional derivative

$$\frac{\delta F[u]}{\delta u(x)} \tag{31}$$

of the following functionals, assuming no variation at the end points. Given $a(x)$ is a known function.

(a)

$$F[u] = \int_{x_1}^{x_2} dx\, a(x) \left[1 + \frac{du(x)}{dx} + \frac{d^2u(x)}{dx^2} + \frac{d^3u(x)}{dx^3}\right] \tag{32}$$

(b)

$$F[u] = \int_a^b dx\, \frac{1}{\left(1 + \dfrac{d^3u}{dx^3}\right)} \tag{33}$$

(c)

$$F[u] = \int_a^b dx\, x^5 \sqrt{1 + \frac{d^3u}{dx^3}} \tag{34}$$

(d)

$$F[u] = \int_a^b dx \sqrt{1 + \frac{du}{dx} + \frac{d^3u}{dx^3}} \tag{35}$$

5. (**20 points.**) Evaluate the functional derivative

$$\frac{\delta W[u]}{\delta u(t)} \tag{36}$$

of the following functionals, with u replaced with the appropriate variable, assuming no variation at the end points.

(a) Let $x(t)$ be position at time t of mass m. The action

$$W[x] = \int_{t_1}^{t_2} dt\, \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 \tag{37}$$

is a functional of position.

(b) Let $z(t)$ be the vertical height at time t of mass m in a uniform gravitational field g. The action

$$W[z] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \left(\frac{dz}{dt}\right)^2 - mgz\right] \tag{38}$$

is a functional of the vertical height.

(c) Let $x(t)$ be the stretch at time t of a spring of spring constant k attached to a mass m. The action

$$W[x] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 - \frac{1}{2} k x^2\right] \tag{39}$$

is a functional of the stretch.

(d) Let $r(t)$ be the radial distance at time t of mass m released from rest in a gravitational field of a planet of mass M. The action

$$W[r] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{GMm}{r}\right] \tag{40}$$

is a functional of the radial distance.

(e) Let $r(t)$ be the radial distance at time t of charge q_1 of mass m released from rest in an electrostatic field of another charge of charge q_2. The action

$$W[r] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \left(\frac{dr}{dt}\right)^2 - \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r}\right] \tag{41}$$

is a functional of the radial distance.