Homework No. 04 (2025 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2025 Feb 11, 4.30pm

1. (**20 points.**) The motion of a particle of mass m near the Earth's surface is described by

$$\frac{d}{dt}(mv) = -mg, \tag{1}$$

where $v = dz/dt$ is the velocity in the upward z direction.

   (a) Find the Lagrangian for this system that implies the equation of motion of Eq. (1) using the principle of stationary action.

   (b) Determine the canonical momentum for this system

   (c) Determine the Hamilton $H(p, z)$ for this system.

   (d) Determine the Hamilton equations of motion.

2. (**20 points.**) The motion of a particle of mass m undergoing simple harmonic motion is described by

$$\frac{d}{dt}(mv) = -kx, \tag{2}$$

where $v = dx/dt$ is the velocity in the x direction.

   (a) Find the Lagrangian for this system that implies the equation of motion of Eq. (2) using the principle of stationary action.

   (b) Determine the canonical momentum for this system

   (c) Determine the Hamiltonian $H(p, x)$ for this system.

   (d) Determine the Hamilton equations of motion.

3. (**20 points.**) (Refer Goldstein, 2nd edition, Chapter 1 Problem 8.) As a consequence of the Hamilton's stationary action principle, the equations of motion for a system can be expressed as Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \tag{3}$$

in terms of a Lagrangian $L(x, \dot{x}, t)$. Show that the Lagrangian for a system is not unique. In particular, show that if $L(x, \dot{x}, t)$ satisfies the Euler-Lagrange equation then

$$L'(x, \dot{x}, t) = L(x, \dot{x}, t) + \frac{dF(x,t)}{dt}, \tag{4}$$

where $F(x, t)$ is any arbitrary differentiable function, also satisfies the Euler-Lagrange equation.

4. (**20 points.**) Consider a (time independent) Hamiltonian

$$H = H(x, p), \tag{5}$$

which satisfies the Hamilton equations of motion

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \tag{6a}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}. \tag{6b}$$

(a) Recollect that the Lagrangian, which will temporarily be called the x-Lagrangian here, is defined by the construction

$$L_x(x, \dot{x}) = p\dot{x} - H(x, p). \tag{7}$$

Starting from Eq. (7) derive

$$\frac{\partial L_x}{\partial x} = -\frac{\partial H}{\partial x}, \tag{8a}$$

$$\frac{\partial L_x}{\partial \dot{x}} = p, \tag{8b}$$

$$\frac{\partial L_x}{\partial p} = \dot{x} - \frac{\partial H}{\partial p}. \tag{8c}$$

Using the Hamilton equations of motion, Eqs. (6), in Eqs. (8) we have the equations governing the x-Lagrangian to be

$$\frac{\partial L_x}{\partial p} = 0, \tag{9a}$$

$$\frac{\partial L_x}{\partial \dot{x}} = p, \tag{9b}$$

$$\frac{d}{dt}\frac{\partial L_x}{\partial \dot{x}} = \frac{\partial L_x}{\partial x}. \tag{9c}$$

(b) Now, define the p-Lagrangian using the construction

$$L_p(p, \dot{p}) = -x\dot{p} - H(x, p). \tag{10}$$

The opposite sign in the construction of the p-Lagrangian is motivated by the action principle, which does not care for a total derivative, refer Schwinger. Starting from Eq. (10) derive

$$\frac{\partial L_p}{\partial p} = -\frac{\partial H}{\partial p}, \tag{11a}$$

$$\frac{\partial L_p}{\partial \dot{p}} = -x, \tag{11b}$$

$$\frac{\partial L_p}{\partial x} = -\dot{p} - \frac{\partial H}{\partial x}. \tag{11c}$$

Using the Hamilton equations of motion, Eqs. (6), in Eqs. (11) we have the equations governing the p-Lagrangian to be

$$\frac{\partial L_p}{\partial x} = 0, \tag{12a}$$

$$\frac{\partial L_p}{\partial \dot{p}} = -x, \tag{12b}$$

$$\frac{d}{dt}\frac{\partial L_p}{\partial \dot{p}} = \frac{\partial L_p}{\partial p}. \tag{12c}$$

(c) Illustrate the above using a specific Hamiltonian, for example that of a harmonic oscillator, as a guide.