Homework No. 14 (2025 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Not applicable

1. (**20 points.**) Relativisitic kinematics is constructed in terms of the proper time element ds, which remains unchanged under a Lorentz transformation,

$$-ds^2 = -c^2 dt^2 + d\mathbf{x} \cdot d\mathbf{x}. \tag{1}$$

Here $\mathbf{x}$ and t are the position and time of a particle. They are components of a vector under Lorentz transformation and together constitute the position four-vector

$$x^\alpha = (ct, \mathbf{x}). \tag{2}$$

(a) Velocity: The four-vector associated with velocity is constructed as

$$u^\alpha = c\frac{dx^\alpha}{ds}. \tag{3}$$

i. Using Eq. (1) deduce

$$\gamma ds = c dt, \quad \text{where} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \quad \mathbf{v} = \frac{d\mathbf{x}}{dt}. \tag{4}$$

Then, show that

$$u^\alpha = (c\gamma, \mathbf{v}\gamma). \tag{5}$$

Here $\mathbf{v}$ is the velocity that we use in Newtonian physics.

ii. Further, show that

$$u^\alpha u_\alpha = -c^2. \tag{6}$$

Thus, conclude that the velocity four-vector is a time-like vector. What is the physical implication of this statement for a particle?

iii. Write down the form of the velocity four-vector in the rest frame of the particle?

(b) Momentum: Define momentum four-vector in terms of the mass m of the particle as

$$p^\alpha = m u^\alpha = (mc\gamma, m\mathbf{v}\gamma). \tag{7}$$

Connection with the physical quantities associated to a moving particle, the energy and momentum of the particle, is made by identifying (or defining)

$$p^\alpha = \left(\frac{E}{c}, \mathbf{p}\right), \tag{8}$$

which corresponds to the definitions

$$E = mc^2\gamma, \tag{9a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{9b}$$

for energy and momentum, respectively. Discuss the non-relativistic limits of these quantities. In particular, using the approximation

$$\gamma = 1 + \frac{1}{2}\frac{v^2}{c^2} + \dots, \tag{10}$$

show that

$$E - mc^2 = \frac{1}{2}mv^2 + \dots, \tag{11a}$$
$$\mathbf{p} = m\mathbf{v} + \dots. \tag{11b}$$

Evaluate

$$p^\alpha p_\alpha = -m^2c^2. \tag{12}$$

Thus, derive the energy-momentum relation

$$E^2 - p^2c^2 = m^2c^4. \tag{13}$$

(c) Acceleration: The four-vector associated with acceleration is constructed as

$$a^\alpha = c\frac{du^\alpha}{ds}. \tag{14}$$

i. Show that

$$a^\alpha = \gamma\left(c\frac{d\gamma}{dt}, \mathbf{v}\frac{d\gamma}{dt} + \gamma\mathbf{a}\right), \tag{15}$$

where

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{16}$$

is the acceleration that we use in Newtonian physics.

ii. Starting from Eq. (6) and taking derivative with respect to proper time show that

$$u^\alpha a_\alpha = 0. \tag{17}$$

Thus, conclude that four-acceleration is space-like.

iii. Further, using the explicit form of $u^\alpha a_\alpha$ in Eq. (17) derive the identity

$$\frac{d\gamma}{dt} = \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\right)\gamma^3. \tag{18}$$

iv. Show that

$$a^\alpha = \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c}\gamma^4, \mathbf{a}\gamma^2 + \frac{\mathbf{v}}{c}\frac{\mathbf{v}\cdot\mathbf{a}}{c}\gamma^4\right) \tag{19}$$

v. Write down the form of the acceleration four-vector in the rest frame ($\mathbf{v} = 0$) of the particle as $(0, \mathbf{a}_0)$, where

$$\mathbf{a}_0 = \mathbf{a}\big|_{\text{rest frame}} \tag{20}$$

is defined as the proper acceleration. Note that the proper acceleration is a Lorentz invariant quantity, that is, independent of which observer makes the measurement.

vi. Evaluate the following identities involving the proper acceleration

$$a^\alpha a_\alpha = \mathbf{a}_0\cdot\mathbf{a}_0 = \left[\mathbf{a}\cdot\mathbf{a} + \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c}\right)^2\gamma^2\right]\gamma^4 = \left[\mathbf{a}\cdot\mathbf{a} - \left(\frac{\mathbf{v}\times\mathbf{a}}{c}\right)^2\right]\gamma^6. \tag{21}$$

vii. In a particular frame, if $\mathbf{v}\,||\,\mathbf{a}$ (corresponding to linear motion), deduce

$$|\mathbf{a}_0| = |\mathbf{a}|\gamma^3. \tag{22}$$

And, in a particular frame, if $\mathbf{v}\perp\mathbf{a}$ (corresponding to circular motion), deduce

$$|\mathbf{a}_0| = |\mathbf{a}|\gamma^2. \tag{23}$$

(d) Force: The force four-vector is defined as

$$f^\alpha = c\frac{dp^\alpha}{ds} = \left(\frac{\gamma}{c}\frac{dE}{dt}, \mathbf{F}\gamma\right), \tag{24}$$

where the force $\mathbf{F}$, identified (or defined) as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \tag{25}$$

is the force in Newtonian physics. Starting from Eq. (12) derive the relation

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v} \tag{26}$$

which is the power output or the rate of work done by the force $\mathbf{F}$ on the particle.

(e) Equations of motion: The relativistic generalization of Newton's laws are

$$f^\alpha = ma^\alpha. \tag{27}$$

Show that these involve the relations, using the definitions of energy and momentum in Eqs. (9),

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}\gamma + m\mathbf{v}\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\gamma^3, \tag{28a}$$

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v} = m\mathbf{v}\cdot\mathbf{a}\gamma^3. \tag{28b}$$

Discuss the non-relativistic limits of the equations of motion.

2. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration α is described by equation of a hyperbola

$$z^2 - c^2t^2 = z_0^2, \qquad z_0 = \frac{c^2}{\alpha}. \tag{29}$$

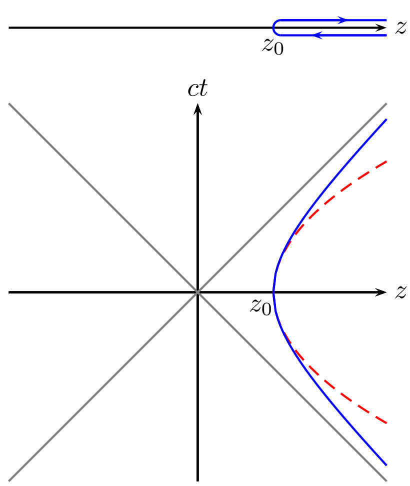

Figure 1: Problem 2

(a) This represents the world-line of a particle thrown from $z > z_0$ at $t < 0$ towards $z = z_0$ in region of constant (proper) acceleration α as described by the bold (blue) curve in the space-time diagram in Figure 2. In contrast a Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$z - z_0 = \frac{1}{2}\alpha t^2 \tag{30}$$

as described by the dashed (red) curve in the space-time diagram in Figure 2. Show that the hyperbolic curve

$$z = z_0\sqrt{1 + \frac{c^2t^2}{z_0^2}} \tag{31}$$

in regions that satisfy

$$t \ll \frac{c}{\alpha} \tag{32}$$

is approximately the parabolic curve

$$z = z_0 + \frac{1}{2}\alpha t^2 + \ldots. \tag{33}$$

(b) Recognize that the proper acceleration α does not have an upper bound.

(c) A large acceleration is achieved by taking an above turn while moving very fast. Thus, turning around while moving close to the speed of light c should achieve the highest acceleration. Show that $\alpha \to \infty$ corresponding to $z_0 \to 0$ represents this scenario. What is the equation of motion of a particle moving with infinite proper acceleration. To gain insight, plot world-lines of particles moving with $\alpha = c^2/z_0$, $\alpha = 10c^2/z_0$, and $\alpha = 100c^2/z_0$.

3. (**20 points.**) A relativisitic particle in a uniform magnetic field is described by the equations

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v}, \tag{34a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{34b}$$

where

$$E = mc^2\gamma, \tag{35a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{35b}$$

and

$$\mathbf{F} = q\mathbf{v}\times\mathbf{B}. \tag{36}$$

Show that

$$\frac{d\gamma}{dt} = 0. \tag{37}$$

Then, derive

$$\frac{d\mathbf{v}}{dt} = \mathbf{v}\times\boldsymbol{\omega}_c, \tag{38}$$

where

$$\boldsymbol{\omega}_c = \frac{q\mathbf{B}}{m\gamma}. \tag{39}$$

Compare this relativistic motion to the associated non-relativistic motion.

4. (**20 points.**) A relativisitic particle in a uniform electric field is described by the equations

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v}, \tag{40a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{40b}$$

where

$$E = mc^2\gamma, \tag{41a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{41b}$$

and

$$\mathbf{F} = q\mathbf{E}. \tag{42}$$

Let us consider the configuration with the electric field in the $\hat{\mathbf{y}}$ direction,

$$\mathbf{E} = E\,\hat{\mathbf{y}}, \tag{43}$$

and initial conditions

$$\mathbf{v}(0) = 0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{44a}$$

$$\mathbf{x}(0) = 0\,\hat{\mathbf{x}} + y_0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}. \tag{44b}$$

(a) In terms of the definition

$$\boldsymbol{\omega}_0 = \frac{1}{c}\frac{q\mathbf{E}}{m}, \tag{45}$$

show that the equations of motion are given by

$$\frac{d\gamma}{dt} = \boldsymbol{\omega}_0\cdot\boldsymbol{\beta} \tag{46}$$

and

$$\frac{d}{dt}(\boldsymbol{\beta}\gamma) = \boldsymbol{\omega}_0. \tag{47}$$

(b) Since the particle starts from rest show that we have

$$\boldsymbol{\beta}\gamma = \boldsymbol{\omega}_0 t. \tag{48}$$

For our configuration this implies

$$\beta_x = 0, \tag{49a}$$

$$\beta_y\gamma = \omega_0 t, \tag{49b}$$

$$\beta_z = 0. \tag{49c}$$

Further, deduce

$$\beta_y = \frac{\omega_0 t}{\sqrt{1+\omega_0^2 t^2}}. \tag{50}$$

Integrate again and use the initial condition to show that the motion is described by

$$y - y_0 = \frac{c}{\bar{\omega}_0}\left[\sqrt{1+\bar{\omega}_0^2 t^2} - 1\right]. \tag{51}$$

Rewrite the solution in the form

$$\left(y - y_0 + \frac{c}{\omega_0}\right)^2 - c^2t^2 = \frac{c^2}{\omega_0^2}. \tag{52}$$

This represents a hyperbola passing through $y = y_0$ at $t = 0$. If we choose the initial position $y_0 = c/\omega_0$ we have

$$y^2 - c^2t^2 = y_0^2. \tag{53}$$

(c) The (constant) proper acceleration associated with this motion is

$$\alpha = \omega_0 c = \frac{c^2}{y_0}. \tag{54}$$

A Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$y - y_0 = \frac{1}{2}\alpha t^2. \tag{55}$$

Show that the hyperbolic curve

$$y = y_0 \sqrt{1 + \frac{c^2 t^2}{y_0^2}} \tag{56}$$

in regions that satisfy

$$\omega_0 t \ll 1 \tag{57}$$

is approximately the parabolic curve

$$y = y_0 + \frac{1}{2}\alpha t^2 + \ldots. \tag{58}$$