Final Exam (Fall 2025)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

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1. (**20 points.**) The magnetic field due to a current on the complete z-axis is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi} \frac{\hat{\boldsymbol{\phi}}}{\rho}, \tag{1}$$

where ϕ is the azimuth angle and ρ is the cylindrical polar coordinate.

(a) Starting from the definition of azimuth angle,

$$\phi = \tan^{-1} \frac{y}{x}, \tag{2}$$

show that

$$\boldsymbol{\nabla}\phi = \frac{\hat{\boldsymbol{\phi}}}{\rho}. \tag{3}$$

(b) Show that

$$\boldsymbol{\nabla} \times \frac{\hat{\boldsymbol{\phi}}}{\rho} = 0, \qquad \text{for} \qquad \rho \neq 0. \tag{4}$$

Then, evaluate

$$\boldsymbol{\nabla} \times \mathbf{B}. \tag{5}$$

(c) Theorem of curl states that

$$\int_S d\mathbf{a} \cdot \boldsymbol{\nabla} \times \mathbf{B} = \oint d\mathbf{l} \cdot \mathbf{B}. \tag{6}$$

Thus, conclude that

$$\boldsymbol{\nabla} \times \frac{\hat{\boldsymbol{\phi}}}{\rho} = \hat{\mathbf{z}}\, 2\pi \, \frac{\delta(\rho)\delta(\phi)}{\rho}. \tag{7}$$

2. (**20 points.**) Evaluate the multipole harmonics of a shell

$$\rho(\mathbf{r}') = \delta(r' - a)\sigma(\theta', \phi') \tag{8}$$

with surface charge density

$$\sigma(\theta', \phi') = \sigma_0 \sin^2\theta' \cos 2\phi'. \tag{9}$$

This is established by determining the non-zero σ_{lm}'s in the expansion

$$\sigma(\theta', \phi') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sigma_{lm} \, Y_{lm}(\theta', \phi') \tag{10}$$

using

$$\sigma_{lm} = \int d\Omega \, Y^*_{lm}(\theta', \phi') \sigma(\theta', \phi'). \tag{11}$$

Hint: Recognize and use

$$\frac{1}{2}\Big[Y_{2,+2}(\theta', \phi') + Y_{2,-2}(\theta', \phi') \Big] = \sqrt{\frac{15}{32\pi}} \sin^2\theta' \cos 2\phi'. \tag{12}$$

Determine the electric potential $\phi(\mathbf{r})$ due to the above charge density using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{13}$$

Using addition fomula for spherical harmonics,

$$P_l(\cos\gamma) = \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta, \phi) \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta', \phi')^*, \tag{14}$$

where

$$\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi'), \tag{15}$$

we learned that

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{1}{r_>} \left(\frac{r_<}{r_>}\right)^l Y_{lm}(\theta, \phi) Y_{lm}(\theta', \phi')^*. \tag{16}$$

Complete the r' integral, Ω' integral, and lm sums, in this order, to show that

$$\phi(\mathbf{r}) = \frac{(4\pi a^2)\, \sigma_0}{4\pi\varepsilon_0} \frac{1}{5} \frac{1}{r_>} \left(\frac{r_<}{r_>}\right)^2 \sin^2\theta \cos 2\phi. \tag{17}$$

3. (**20 points.**) A mass m experiencing a linear drag, in addition to a time-dependent force $F(t)$, in one-dimension is described by the differential equation

$$ma = -\gamma v + F(t), \tag{18}$$

with initial conditions

$$v(0) = 0 \qquad \text{and} \qquad F(0) \neq 0. \tag{19}$$

where v is velocity, a is acceleration, and γ is a material dependent, fluid dependent, and interface geometry dependent, parameter. Show that the above differential equation can be expressed in the form

$$\left(\frac{d}{dt}+b\right)v(t)=h(t), \tag{20}$$

where $b=\gamma/m$ and $h(t)=F(t)/m$. The Green function for the above differential equation is

$$\left(\frac{d}{dt}+b\right)g(t,t')=\delta(t-t'). \tag{21}$$

(a) Integrate the differential equation to derive the continuity condition

$$g(t,t')\Big|_{t=t'-\delta}^{t=t'+\delta}+b\int_{t'-\delta}^{t'+\delta}dt\,g(t,t')=1 \tag{22}$$

which is realized by requiring the discontinuity in the Green function to be

$$g(t,t')\Big|_{t=t'-\delta}^{t=t'+\delta}=1. \tag{23}$$

Discuss the nature of discontinuities for which Eq. (23) implies

$$\int_{t'-\delta}^{t'+\delta}dt\,g(t,t')=0. \tag{24}$$

(b) We have homogeneous differential equations for $t\neq t'$ who solutions can be expressed in terms of arbitray constant A and B as

$$g(t,t')=\begin{cases}A\,e^{-bt}, & t<t',\\ B\,e^{-bt}, & t'<t.\end{cases} \tag{25}$$

Show that a particular solution is

$$g(t,t')=\theta(t-t')\,e^{-b(t-t')}, \tag{26}$$

where is the Heaviside step function. A particular solution is off by a homogeneous solution, e^{-bt}. Verify, by substitution in the original differential equation, that

$$g(t,t')=A\,e^{-bt}+\theta(t-t')\,e^{-b(t-t')}, \tag{27}$$

is also a solution.

(c) The solution for velocity, upto a homogeneous solution, is given by

$$v(t)=\int_{-\infty}^{\infty}dt'g(t,t')h(t'). \tag{28}$$

Thus, show that

$$v(t) = D\, e^{-bt} + \int_{-\infty}^{t} dt' e^{-b(t-t')} h(t') \tag{29}$$

is general solution where D is determined using he initial condition on velocity. For $v(0) = 0$ show that

$$D = -\int_{-\infty}^{0} dt' e^{-b(t-t')} h(t') \tag{30}$$

and the solution has the form

$$v(t) = \int_{0}^{t} dt' e^{-b(t-t')} h(t'). \tag{31}$$

(d) Find the expression for velocity for a uniform force $h(t) = g$. What is the terminal velocity in this scenario?

4. (**20 points.**) The following recording available at

https://www.youtube.com/watch?v=D97Liq4In2A&t=6540

is a resource. The Green function for a wave equation is

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r} - \mathbf{r}', t - t') = \delta^{(3)}(\mathbf{r} - \mathbf{r}')\delta(t - t'). \tag{32}$$

(a) Let $\mathbf{r}' = 0$ and $t' = 0$. Then, Fourier transform in time to obtain

$$-\left(\nabla^2 + \frac{\omega^2}{c^2}\right) G(\mathbf{r}; \omega) = \delta^{(3)}(\mathbf{r}), \tag{33}$$

for a particular mode of frequency ω.

(b) Integrate around the source at $\mathbf{r}'$ to obtain the continuity condition

$$\lim_{r\to 0}(4\pi r^2)\, \hat{\mathbf{r}} \cdot \boldsymbol{\nabla} G = -1. \tag{34}$$

(c) Integrate the angular part, use spherical symmetry, to express the differential equation as

$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\omega^2}{c^2}\right) G(r; \omega) = \frac{\delta(r)}{4\pi r^2} \tag{35}$$

and rewrite the continuity condition in the form

$$\lim_{r\to 0} r^2 \frac{\partial G}{\partial r} = -\frac{1}{4\pi}. \tag{36}$$

(d) In the static limit, $\omega \to 0$, the Green function reduces to

$$\lim_{\omega \to 0} G(r;\omega) = \frac{1}{4\pi r}. \tag{37}$$

Thus, define $g(r;\omega)$ using

$$G(r;\omega) = \frac{g(r;\omega)}{4\pi r} \tag{38}$$

and show that it satisfies the differential equation

$$-\left(\frac{d^2}{dr^2} + \frac{\omega^2}{c^2}\right) g(r;\omega) = \frac{\delta(r)}{r} \tag{39}$$

with continuity condition

$$\lim_{r \to 0} g(r;\omega) = 1. \tag{40}$$

(e) Solve for $g(r;\omega)$ and find

$$g(r;\omega) = Ae^{i\frac{\omega}{c}r} + Be^{-i\frac{\omega}{c}r} \tag{41}$$

with the constraint

$$A + B = 1. \tag{42}$$

Thus, show that

$$G(\mathbf{r}-\mathbf{r}';\omega) = \frac{Ae^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{Be^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{43}$$

Fourier transform to show that

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{A\delta(t-t'-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{B\delta(t-t'+\frac{1}{c}|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{44}$$

Requiring the Green function to be causal, that is, $t > t'$, show that $A = 1$ and $B = 0$.

5. (**20 points.**) Given

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{\delta(t-t'-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|}, \tag{45}$$

evaluate the integral

$$\int_{-\infty}^{\infty} dt\, G(\mathbf{r}-\mathbf{r}', t-t'). \tag{46}$$

From the answer what can you comment about the physical interpretation of $\int_{-\infty}^{\infty} dt\, G$?