Midterm Exam No. 01 (Fall 2025)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Date: 2025 Sep 26

1. (**20 points.**) Let the rectangular components of position vector $\mathbf{r}$ be represented by r_i and the rectangular components of momentum vector $\mathbf{p}$ be represented by p_i. In Newtonian mechanics these components satisfy the commutation relations
$$\big[r_i, p_j\big] = 0. \tag{1}$$
Then, show that
$$(\mathbf{r} \times \mathbf{p}) \cdot \mathbf{r} = 0. \tag{2}$$
In quantum mechanics governed by Heisenberg equations of motion the components of position and momentum vectors satisfy the commutation relations
$$\big[r_i, p_j\big] = i\hbar\, \delta_{ij}, \tag{3}$$
$\hbar = h/(2\pi)$, where h is the Planck constant. Will the vector identity in Eq. (2) be satisfied in quantum mechanics? If not, determine the modified relation.

2. (**20 points.**) Evaluate
$$\nabla^2 \left(\frac{1}{\mathbf{a} \cdot \mathbf{r}} \right), \tag{4}$$
where $\mathbf{a}$ is a constant vector.

3. (**20 points.**) The eigenbasis for rectangular coordinates on a two-dimensional plane satisfy the completeness relation
$$\mathbf{1} = \hat{\mathbf{i}}\,\hat{\mathbf{i}} + \hat{\mathbf{j}}\,\hat{\mathbf{j}} \tag{5}$$
and the eigenbasis for polar coordinates on a plane satisfy the completeness relation
$$\mathbf{1} = \hat{\mathbf{r}}\,\hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}\,\hat{\boldsymbol{\phi}}, \tag{6}$$
where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ are radial and tangential unit vectors, respectively. A transformation operator that connects the above two eigenbases has the dyadic construction
$$\mathbf{T} = \hat{\mathbf{r}}\,\hat{\mathbf{i}} + \hat{\boldsymbol{\phi}}\,\hat{\mathbf{j}}. \tag{7}$$
Evaluate the following:
$$\mathbf{T} \cdot \hat{\mathbf{i}} = \tag{8a}$$
$$\mathbf{T} \cdot \hat{\mathbf{j}} = \tag{8b}$$

4. (**20 points. Take home.**) Let

$$f(z) = \frac{(z-2)(z+2)}{z(z-2i)(z+2i)}. \tag{9}$$

   (a) Find the zeros of $f(z)$.

   (b) Find the poles of $f(z)$.

   Remember to include 'the point at infinity' in your analysis.

5. (**20 points.**) Evaluate the contour integral

$$I(a) = \frac{1}{2\pi i} \oint_c dz \frac{a e^{az}}{az-1}, \tag{10}$$

   where the contour c is a unit circle going counterclockwise with center at the origin. Presume a is complex number outside the contour, that is, $|a| > 1$.