(Preview of) Midterm Exam No. 02 (Fall 2025)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Date: 2025 Nov 7

1. (**20 points.**) On contents related to HW-06, tentatively on probabilities.
2. (**20 points.**) On contents related to HW-07, tentatively on Fourier transform.
3. (**20 points.**) On contents related to HW-08, tentatively on Green's function.
4. (**20 points.**) On contents related to HW-09, tentatively on Lecture dated 20251031.
5. (**20 points.**) Spherical harmonics or surface harmonics of degree l are

$$Y_l(\mathbf{r}) = \sqrt{\frac{2l+1}{4\pi}} \frac{r^{l+1}}{l!} (-\mathbf{s}_1 \cdot \boldsymbol{\nabla})(-\mathbf{s}_2 \cdot \boldsymbol{\nabla}) \ldots (-\mathbf{s}_l \cdot \boldsymbol{\nabla}) \frac{1}{r} \tag{1}$$

for $l = 0, 1, 2, \ldots,$ with

$$Y_0(\mathbf{r}) = \sqrt{\frac{1}{4\pi}}, \tag{2}$$

where $\mathbf{s}_l$ are constant vectors. Here constant refers to independent of position $\mathbf{r}$. Recall that the fundamental solution to Laplace's equation is the electric potential due to a point charge,

$$\frac{q}{4\pi\varepsilon_0} \frac{1}{r}. \tag{3}$$

Dropping $q/(4\pi\varepsilon_0)$ we have

$$\nabla^2 \frac{1}{r} = 0, \quad r \neq 0, \tag{4}$$

where $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. Zonal harmonics $P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{z}})$ of degree l are defined in terms of spherical harmonics of degree l for the choice

$$\mathbf{s}_1 = \mathbf{s}_2 = \cdots = \mathbf{s}_l = \hat{\mathbf{z}}, \tag{5}$$

as

$$P_l(\hat{\mathbf{r}} \cdot \hat{\mathbf{z}}) = \sqrt{\frac{4\pi}{2l+1}} Y_l(\hat{\mathbf{r}}). \tag{6}$$

In terms of

$$\mu = \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = \hat{\mathbf{z}} \cdot \boldsymbol{\nabla} r = \frac{\partial r}{\partial z} = \frac{z}{r} \tag{7}$$

show that

$$P_l(\mu) = \frac{r^{l+1}}{l!} \left(-\frac{\partial}{\partial z} \right)^l \frac{1}{r}. \tag{8}$$

Evaluate the zonal harmonics of degree $l = 0, 1, 2, 3$.