Homework No. 01 (Fall 2025)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2025 Aug 25, 4.30pm

1. (**20 points.**) Verify the following relations:

$$\begin{aligned} \delta_{ij} &= \delta_{ji}, & \text{(1a)} \\ \delta_{ii} &= 3, & \text{(1b)} \\ \delta_{ik}\delta_{kj} &= \delta_{ij}, & \text{(1c)} \\ \delta_{im}B_m &= B_i, & \text{(1d)} \\ \varepsilon_{ijk} &= -\varepsilon_{ikj} = \varepsilon_{kij}, & \text{(1e)} \\ \varepsilon_{iik} &= 0, & \text{(1f)} \\ \delta_{ij}\varepsilon_{ijk} &= 0. & \text{(1g)} \end{aligned}$$

2. (**20 points.**) In three dimensions the Levi-Civita symbol is given in terms of the determinant of the Kronecker δ-functions,

$$\begin{aligned} \varepsilon_{ijk}\varepsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} & \text{(2a)} \\ &= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}) + \delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}). & \text{(2b)} \end{aligned}$$

Using the above identity show that

$$\begin{aligned} \varepsilon_{ijk}\varepsilon_{imn} &= \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}, & \text{(3a)} \\ \varepsilon_{ijk}\varepsilon_{ijn} &= 2\delta_{kn}, & \text{(3b)} \\ \varepsilon_{ijk}\varepsilon_{ijk} &= 6. & \text{(3c)} \end{aligned}$$

3. (**20 points.**) Given that the rectangular components of vector $\mathbf{A}$, represented by A_i, and the rectangular components of vector $\mathbf{B}$, represented by B_i, satisfy the commutation relations

$$\big[A_i, B_j\big] = i\hbar\,\delta_{ij}, \tag{4}$$

$\hbar = h/(2\pi)$, where h is the Planck constant. Using index notation and the properties of Kronecker δ-function express the left hand side of the following equation in the form on the right hand side,

$$\mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A} = i\hbar\,\alpha. \tag{5}$$

Find α. Investigate the case when $\hbar = 0$.

4. (**20 points.**) Given that the rectangular components of vector $\mathbf{L}$, represented by L_i, satisfy the commutation relations

$$\big[L_i, L_j\big] = i\hbar\, \varepsilon_{ijk} L_k, \tag{6}$$

$\hbar = h/(2\pi)$, where h is the Planck constant. Using index notation and the properties of Kronecker δ-function and Levi-Civita symbol, show that

$$\mathbf{L} \times \mathbf{L} = i\hbar\, \mathbf{L}. \tag{7}$$

Investigate the case when $\hbar = 0$.

5. (**20 points.**) Given that the rectangular components of vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, represented by A_i, B_i, and C_i, respectively, satisfy the commutation relations

$$\big[A_i, B_j\big] = i\hbar\, \delta_{ij}, \tag{8a}$$
$$\big[B_i, C_j\big] = 0, \tag{8b}$$
$$\big[C_i, A_j\big] = 0, \tag{8c}$$

$\hbar = h/(2\pi)$, where h is the Planck constant. Using index notation and the properties of Kronecker δ-function and Levi-Civita symbol, show that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \alpha\, \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \beta\, \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + i\hbar\, \gamma\, \mathbf{C}. \tag{9}$$

Find α, β, and γ. Investigate the case when $\hbar = 0$.

6. (**20 points.**) Given that the rectangular components of vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$, represented by A_i, B_i, C_i, and D_i, respectively, satisfy the commutation relations

$$\big[A_i, B_j\big] = i\hbar\, \delta_{ij}, \qquad \big[B_i, C_j\big] = i\hbar\, \delta_{ij}, \tag{10a}$$
$$\big[A_i, C_j\big] = i\hbar\, \delta_{ij}, \qquad \big[B_i, D_j\big] = i\hbar\, \delta_{ij}, \tag{10b}$$
$$\big[A_i, D_j\big] = i\hbar\, \delta_{ij}, \qquad \big[C_i, D_j\big] = i\hbar\, \delta_{ij}, \tag{10c}$$

$\hbar = h/(2\pi)$, where h is the Planck constant. Using index notation and the properties of Kronecker δ-function and Levi-Civita symbol, show that

$$(\mathbf{A} \times \mathbf{B})(\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) + (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) - i\hbar\, \mathbf{A} \cdot (\mathbf{B} + \mathbf{C} - \mathbf{D}). \tag{11}$$

Investigate the case when $\hbar = 0$.