Homework No. 02 (Fall 2025)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Friday, 2025 Sep 5, 4.30pm

1. (**20 points.**) Show that, if $\mathbf{r}$ and $\mathbf{p}$ are Hermitian, the operator construction for angular momentum
$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{1}$$
is also Hermitian. Show that the symmetric construction
$$\mathbf{L} = \frac{1}{2}\Big[(\mathbf{r} \times \mathbf{p}) + (\mathbf{r} \times \mathbf{p})^\dagger\Big] \tag{2}$$
is also Hermitian. Evaluate the measure of non-commutativity of the angular momentum operator by evaluating
$$\frac{1}{2}\Big[(\mathbf{r} \times \mathbf{p}) - (\mathbf{r} \times \mathbf{p})^\dagger\Big]. \tag{3}$$

2. (**20 points.**) The components of the position and momentum operator, $\mathbf{r}$ and $\mathbf{p}$, respectively, satisfy the commutation relations $[r_i, p_j] = i\hbar\delta_{ij}$. Verify the following:
   (a) $\mathbf{r} \times \mathbf{p} + \mathbf{p} \times \mathbf{r} = 0$.
   (b) $\mathbf{r} \cdot \mathbf{p} - \mathbf{p} \cdot \mathbf{r} = 3i\hbar$.
   (c) $(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{p}) - (\mathbf{b} \cdot \mathbf{p})(\mathbf{a} \cdot \mathbf{r}) = i\hbar(\mathbf{a} \cdot \mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ and numerical.
   (d) $\mathbf{r} \times (\mathbf{r} \times \mathbf{p}) = \mathbf{r}\,\mathbf{p} \cdot \mathbf{r} - \mathbf{p}r^2 + i\hbar\mathbf{r}$.

3. (**20 points.**) Use index notation or dyadic notation to show that
$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2\mathbf{A}, \tag{4a}$$
$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \mathbf{B}), \tag{4b}$$
$$\boldsymbol{\nabla} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{A} + \mathbf{A}(\boldsymbol{\nabla} \cdot \mathbf{B}) - (\boldsymbol{\nabla} \cdot \mathbf{A})\mathbf{B} - (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{B}. \tag{4c}$$

4. (**20 points.**) Consider the dyadic construction
$$\mathbf{T} = \mathbf{E}\,\mathbf{B} \tag{5}$$
built using the vector fields,
$$\mathbf{E} = E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}}, \tag{6a}$$
$$\mathbf{B} = B\,\hat{\mathbf{y}}. \tag{6b}$$

Evaluate the following components of the dyadic:

$$\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} = \tag{7a}$$
$$\hat{\mathbf{y}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \hat{\mathbf{y}} \cdot \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \hat{\mathbf{y}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} = \tag{7b}$$
$$\hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} = \tag{7c}$$

Evaluate the scalars

$$\mathrm{Tr}(\mathbf{T}) = T_{ii}, \tag{8a}$$
$$\mathrm{Tr}(\mathbf{T} \cdot \mathbf{T}) = T_{ij} T_{ji}, \tag{8b}$$
$$\mathrm{Tr}(\mathbf{T} \cdot \mathbf{T} \cdot \mathbf{T}) = T_{ij} T_{jk} T_{ki}. \tag{8c}$$

Evaluate the following vector field constructions:

$$\hat{\mathbf{x}} \cdot \mathbf{T} = \qquad \hat{\mathbf{y}} \cdot \mathbf{T} = \qquad \hat{\mathbf{z}} \cdot \mathbf{T} = \tag{9a}$$
$$\mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \mathbf{T} \cdot \hat{\mathbf{z}} = \tag{9b}$$
$$\hat{\mathbf{x}} \times \mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \hat{\mathbf{y}} \times \mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \hat{\mathbf{z}} \times \mathbf{T} \cdot \hat{\mathbf{x}} = \tag{9c}$$
$$\hat{\mathbf{x}} \times \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \hat{\mathbf{y}} \times \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \hat{\mathbf{z}} \times \mathbf{T} \cdot \hat{\mathbf{y}} = \tag{9d}$$
$$\hat{\mathbf{x}} \times \mathbf{T} \cdot \hat{\mathbf{z}} = \qquad \hat{\mathbf{y}} \times \mathbf{T} \cdot \hat{\mathbf{z}} = \qquad \hat{\mathbf{z}} \times \mathbf{T} \cdot \hat{\mathbf{z}} = \tag{9e}$$
$$\hat{\mathbf{x}} \cdot \mathbf{T} \times \hat{\mathbf{x}} = \qquad \hat{\mathbf{x}} \cdot \mathbf{T} \times \hat{\mathbf{y}} = \qquad \hat{\mathbf{x}} \cdot \mathbf{T} \times \hat{\mathbf{z}} = \tag{9f}$$
$$\hat{\mathbf{y}} \cdot \mathbf{T} \times \hat{\mathbf{x}} = \qquad \hat{\mathbf{y}} \cdot \mathbf{T} \times \hat{\mathbf{y}} = \qquad \hat{\mathbf{y}} \cdot \mathbf{T} \times \hat{\mathbf{z}} = \tag{9g}$$
$$\hat{\mathbf{z}} \cdot \mathbf{T} \times \hat{\mathbf{x}} = \qquad \hat{\mathbf{z}} \cdot \mathbf{T} \times \hat{\mathbf{y}} = \qquad \hat{\mathbf{z}} \cdot \mathbf{T} \times \hat{\mathbf{z}} = \tag{9h}$$

5. (**20 points.**) Verify the following identities:

$$\boldsymbol{\nabla} r = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}, \tag{10a}$$
$$\boldsymbol{\nabla} \mathbf{r} = \mathbf{1}. \tag{10b}$$

Further, show that

$$\boldsymbol{\nabla} \cdot \mathbf{r} = 3, \tag{11a}$$
$$\boldsymbol{\nabla} \times \mathbf{r} = 0. \tag{11b}$$

Here r is the magnitude of the position vector $\mathbf{r}$, and $\hat{\mathbf{r}}$ is the unit vector pointing in the direction of $\mathbf{r}$.

6. (**20 points.**) Evaluate the left hand side of the equation

$$\boldsymbol{\nabla}(\mathbf{r} \cdot \mathbf{p}) = a\,\mathbf{p} + b\,\mathbf{r}, \tag{12}$$

where $\mathbf{p}$ is a constant vector. Thus, find a and b.

7. (**20 points.**) Evaluate

$$\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{r}}{r^3} \right), \tag{13}$$

everywhere in space, including $\mathbf{r} = 0$.
Hint: Check your answer for consistency by using divergence theorem.

8. (**20 points.**) Evaluate

$$\boldsymbol{\nabla} \left(\frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right), \tag{14}$$

where $\mathbf{p}$ is a constant vector.

9. (**20 points.**) Consider the distribution

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}. \tag{15}$$

Show that

$$\delta(x) \begin{cases} \to \infty, & \text{if} \quad x = 0, \\ \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{16}$$

Further, show that

$$\int_{-\infty}^{\infty} dx \, \delta(x) = 1. \tag{17}$$

Plot $\delta(x)$ before taking the limit $\varepsilon \to 0$ and identify ε in the plot.

10. (**10 points.**) A uniformly charged infinitely thin disc of radius R and total charge Q is placed on the x-y plane such that the normal vector is along the z axis and the center of the disc at the origin. Write down the charge density of the disc in terms of δ-function(s). Integrate over the charge density and verify that it returns the total charge on the disc.

11. (**10 points.**) An (idealized) infinitely long wire, (on the z-axis with infinitesimally small cross sectional area,) carrying a current I can be mathematically represented by the current density

$$\mathbf{J}(\mathbf{x}) = \hat{\mathbf{z}} \, I \, \delta(x) \delta(y). \tag{18}$$

A similar idealized wire forms a circular loop and is placed on the xy-plane with the center of the circular loop at the origin. Write down the current density of the circular loop carrying current I.