Homework No. 03 (Fall 2025)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2025 Sep 8, 4.30pm

0. Problems 1 and 3 are for practice. Problem 2 is for submission.

1. (**Example.**) Let $\mathbf{r}$ represent the position vector, x^i the components of the position vector in rectangular coordinates, and u^i the components of the position vector in cylindrical polar coordinates. In particular, we have

$$x^1 = x = \rho\cos\phi, \qquad u^1 = \rho = \sqrt{x^2+y^2}, \tag{1a}$$

$$x^2 = y = \rho\sin\phi, \qquad u^2 = \phi = \tan^{-1}\frac{y}{x}, \tag{1b}$$

$$x^3 = z = z, \qquad u^3 = z = z. \tag{1c}$$

Let us define the unit vectors

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2a}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2b}$$

$$\hat{\mathbf{z}} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + \hat{\mathbf{k}}, \tag{2c}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are basis vectors in rectangular coordinate system. We will also use the notation $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}^3 = \hat{\mathbf{x}}_3$, to represent these vectors.

   (a) Basis vectors:

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i}. \tag{3}$$

   Show that

$$\mathbf{e}_1 = \hat{\boldsymbol{\rho}}, \qquad \mathbf{e}_2 = \rho\hat{\boldsymbol{\phi}}, \qquad \mathbf{e}_3 = \hat{\mathbf{z}}. \tag{4}$$

   (b) Reciprocal basis vectors:

$$\mathbf{e}^i = \boldsymbol{\nabla} u^i. \tag{5}$$

   Show that

$$\mathbf{e}^1 = \hat{\boldsymbol{\rho}}, \qquad \mathbf{e}^2 = \frac{\hat{\boldsymbol{\phi}}}{\rho}, \qquad \mathbf{e}^3 = \hat{\mathbf{z}}. \tag{6}$$

   (c) Orthonormality: Show that

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j. \tag{7}$$

(d) Metric tensor: A line element is defined as

$$d\mathbf{r} = dx^i \hat{\mathbf{x}}_i = du^i \mathbf{e}_i. \tag{8}$$

Show that

$$d\mathbf{r} \cdot d\mathbf{r} = du^i du^j g_{ij}, \tag{9}$$

where the metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \tag{10}$$

Evaluate all the components of g_{ij}.

(e) Completeness relation: Starting from

$$\boldsymbol{\nabla}\, \mathbf{r} = \mathbf{1} \tag{11}$$

derive the completeness relation

$$\mathbf{e}^i \, \mathbf{e}_i = \mathbf{1}. \tag{12}$$

Express the completeness relation in cylindrical polar coordinates in terms of $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$.

2. (**20 points.**) Transformation of basis vectors.

(a) Let us consider a set of basis vectors $\mathbf{e}_i$, where $i = 1, 2, 3$, and the associated reciprocal basis vectors $\mathbf{e}^i$ that satisfy the completeness relation

$$\mathbf{1} = \mathbf{e}^i \, \mathbf{e}_i = \mathbf{e}^1 \mathbf{e}_1 + \mathbf{e}^2 \mathbf{e}_2 + \mathbf{e}^3 \mathbf{e}_3. \tag{13}$$

It is a complete set because an arbitrary vector $\mathbf{A}$ can be expressed in terms of its projections along the basis vectors in the following way,

$$\mathbf{A} = \mathbf{A} \cdot \mathbf{1} = \mathbf{A} \cdot (\mathbf{e}^i \, \mathbf{e}_i) = (\mathbf{A} \cdot \mathbf{e}^i) \, \mathbf{e}_i = A^i \, \mathbf{e}_i, \tag{14}$$

where we recognized and defined the projections of vector $\mathbf{A}$ along the direction of basis vectors as the components

$$A^i = (\mathbf{A} \cdot \mathbf{e}^i). \tag{15}$$

Similarly, multiplying by the identity dyadic on the left gives

$$\mathbf{A} = \mathbf{1} \cdot \mathbf{A} = (\mathbf{e}^i \, \mathbf{e}_i) \cdot \mathbf{A} = \mathbf{e}^i \, (\mathbf{e}_i \cdot \mathbf{A}) = \mathbf{e}^i \, A_i, \tag{16}$$

where now the projections of vector $\mathbf{A}$ in the direction of the reciprocal basis vectors are the components

$$A_i = (\mathbf{e}_i \cdot \mathbf{A}). \tag{17}$$

For consistency we require the equality

$$(\mathbf{A} \cdot \mathbf{e}^i) \, \mathbf{e}_i = \mathbf{e}^i \, (\mathbf{e}_i \cdot \mathbf{A}). \tag{18}$$

Thus, derive

$$A^j = g^{ji} A_i, \tag{19a}$$
$$A_j = g_{ji} A^i, \tag{19b}$$

where the metric tensors are defined as where the metric tensors are defined as

$$g^{ji} = \mathbf{e}^j \cdot \mathbf{e}^i, \tag{20a}$$
$$g_{ji} = \mathbf{e}_j \cdot \mathbf{e}_i. \tag{20b}$$

(b) For another set of basis vectors $\mathbf{g}_i$ and the associated reciprocal basis vectors $\mathbf{g}^i$ that satisfy the completeness relation

$$\mathbf{1} = \mathbf{g}^i \, \mathbf{g}_i \tag{21}$$

we can write

$$\mathbf{A} = (\mathbf{A} \cdot \mathbf{g}^i) \, \mathbf{g}_i = \bar{A}^i \, \mathbf{g}_i, \tag{22}$$

where the components $\bar{A}^i$ are in general different from A^i, and

$$\mathbf{A} = \mathbf{g}^i \, (\mathbf{g}_i \cdot \mathbf{A}) = \mathbf{g}^i \, \bar{A}_i. \tag{23}$$

For consistency we require the equality

$$(\mathbf{A} \cdot \mathbf{g}^i) \, \mathbf{g}_i = \mathbf{g}^i \, (\mathbf{g}_i \cdot \mathbf{A}). \tag{24}$$

Thus, derive

$$\bar{A}^j = \bar{g}^{ji} \bar{A}_i, \tag{25a}$$
$$\bar{A}_j = \bar{g}_{ji} \bar{A}^i, \tag{25b}$$

where the metric tensors are defined as

$$\bar{g}^{ji} = \mathbf{g}^j \cdot \mathbf{g}^i, \tag{26a}$$
$$\bar{g}_{ji} = \mathbf{g}_j \cdot \mathbf{g}_i. \tag{26b}$$

(c) For consistency between the two independent basis vector representations we require the equality

$$(\mathbf{A} \cdot \mathbf{e}^i) \, \mathbf{e}_i = (\mathbf{A} \cdot \mathbf{g}^i) \, \mathbf{g}_i. \tag{27}$$

Taking the dot product on the right with $\mathbf{e}^j$ and using orthogonality relation

$$\mathbf{e}_i \cdot \mathbf{e}^i = \delta^i_j \tag{28}$$

we obtain

$$(\mathbf{A} \cdot \mathbf{e}^j) = (\mathbf{A} \cdot \mathbf{g}^i) \, [\mathbf{g}_i \cdot \mathbf{e}^j]. \tag{29}$$

Similarly, taking the dot product on the right with $\mathbf{g}^j$ we obtain

$$(\mathbf{A} \cdot \mathbf{e}^i)\,[\mathbf{e}_i \cdot \mathbf{g}^j] = (\mathbf{A} \cdot \mathbf{g}^j). \tag{30}$$

In terms of the transformation matrices connecting the two basis vectors,

$$S_i{}^j = [\mathbf{g}_i \cdot \mathbf{e}^j] \tag{31}$$

and

$$R_i{}^j = [\mathbf{e}_i \cdot \mathbf{g}^j], \tag{32}$$

we can derive the transformation of the components

$$A^j = \bar{A}^i S_i{}^j, \tag{33a}$$

$$A^i R_i{}^j = \bar{A}^j. \tag{33b}$$

Further, derive

$$A_j = R_j{}^i \bar{A}_i, \tag{34a}$$

$$S_j{}^i A_i = \bar{A}^j. \tag{34b}$$

Show that

$$R_i{}^j S_j{}^k = [\mathbf{e}_i \cdot \mathbf{g}^j][\mathbf{g}_j \cdot \mathbf{e}^k] = \mathbf{e}_i \cdot (\mathbf{g}^j\,\mathbf{g}_j) \cdot \mathbf{e}^k = \mathbf{e}_i \cdot \mathbf{1} \cdot \mathbf{e}^k = \delta_i^k. \tag{35}$$

(d) i. Find the transformation matrices S and T between cylindrical polar coordinates and rectangular coordinates. Verify that $ST = 1$.

ii. Find the transformation matrices S and T between cylindrical polar coordinates and spherical polar coordinates. Verify that $ST = 1$.

3. (**Example.**) Let $\mathbf{r}$ represent a position vector in three dimensional space. Let x^i be the components of the position vector in rectangular coordinates, which can be interpreted as surfaces of constant x^i. Let us coordinatize the space using the planes, labeled using β,

$$y = mx + \beta \tag{36}$$

where m is fixed, instead of planes with constant y. The other two sets of planes of constant x and constant z are the same. See Fig. 1. Let u^i be the components of the position vector in this new coordinatization of space. In particular, we have

$$x^1 = x = \alpha, \qquad u^1 = \alpha = x, \tag{37a}$$

$$x^2 = y = mx + \beta, \qquad u^2 = \beta = y - mx, \tag{37b}$$

$$x^3 = z = \gamma, \qquad u^3 = \gamma = z. \tag{37c}$$

The basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ in rectangular coordinate system will be represented as $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}^3 = \hat{\mathbf{x}}_3$, if necessary.

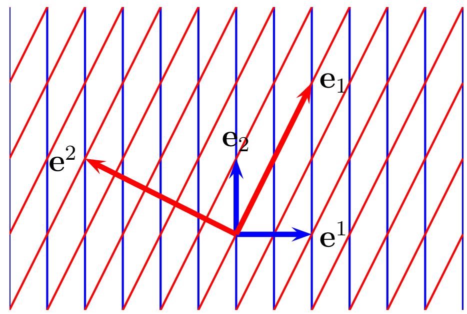

Figure 1: Basis vectors $\mathbf{e}_i$ and reciprocal basis vectors $\mathbf{e}^i$.

(a) Basis vectors:

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i}. \tag{38}$$

Show that

$$\mathbf{e}_1 = \hat{\mathbf{i}} + m\,\hat{\mathbf{j}}, \qquad \mathbf{e}_2 = \hat{\mathbf{j}}, \qquad \mathbf{e}_3 = \hat{\mathbf{k}}. \tag{39}$$

(b) Reciprocal basis vectors:

$$\mathbf{e}^i = \boldsymbol{\nabla} u^i. \tag{40}$$

Show that

$$\mathbf{e}^1 = \hat{\mathbf{i}}, \qquad \mathbf{e}^2 = -m\,\hat{\mathbf{i}} + \hat{\mathbf{j}}, \qquad \mathbf{e}^3 = \hat{\mathbf{k}}. \tag{41}$$

Verify the relations

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{(\mathbf{e}_2 \times \mathbf{e}_3)\cdot \mathbf{e}_1}, \qquad \mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{(\mathbf{e}_3 \times \mathbf{e}_1)\cdot \mathbf{e}_2}, \qquad \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{(\mathbf{e}_1 \times \mathbf{e}_2)\cdot \mathbf{e}_3}. \tag{42}$$

(c) Orthonormality: Show that

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j. \tag{43}$$

That is,

$$\mathbf{e}^1 \cdot \mathbf{e}_1 = 1, \qquad \mathbf{e}^1 \cdot \mathbf{e}_2 = 0, \qquad \mathbf{e}^1 \cdot \mathbf{e}_3 = 0, \tag{44a}$$

$$\mathbf{e}^2 \cdot \mathbf{e}_1 = 0, \qquad \mathbf{e}^2 \cdot \mathbf{e}_2 = 1, \qquad \mathbf{e}^2 \cdot \mathbf{e}_3 = 0, \tag{44b}$$

$$\mathbf{e}^3 \cdot \mathbf{e}_1 = 0, \qquad \mathbf{e}^3 \cdot \mathbf{e}_2 = 0, \qquad \mathbf{e}^3 \cdot \mathbf{e}_3 = 1. \tag{44c}$$

(d) Metric tensor: The metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \tag{45}$$

Evaluate all the components of g_{ij}. That is,

$$g_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1 + m^2, \qquad g_{12} = \mathbf{e}_1 \cdot \mathbf{e}_2 = m, \qquad g_{13} = \mathbf{e}_1 \cdot \mathbf{e}_3 = 0, \tag{46a}$$

$$g_{21} = \mathbf{e}_2 \cdot \mathbf{e}_1 = m, \qquad g_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1, \qquad g_{23} = \mathbf{e}_2 \cdot \mathbf{e}_3 = 0, \tag{46b}$$

$$g_{31} = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0, \qquad g_{32} = \mathbf{e}_3 \cdot \mathbf{e}_2 = 0, \qquad g_{33} = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1. \tag{46c}$$

Similarly evaluate the components of

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j. \tag{47}$$

That is,

$$g^{11} = \mathbf{e}^1 \cdot \mathbf{e}^1 = 1, \qquad g^{12} = \mathbf{e}^1 \cdot \mathbf{e}^2 = -m, \qquad g^{13} = \mathbf{e}^1 \cdot \mathbf{e}^3 = 0, \tag{48a}$$
$$g^{21} = \mathbf{e}^2 \cdot \mathbf{e}^1 = -m, \qquad g^{22} = \mathbf{e}^2 \cdot \mathbf{e}^2 = 1 + m^2, \qquad g^{23} = \mathbf{e}^2 \cdot \mathbf{e}^3 = 0, \tag{48b}$$
$$g^{31} = \mathbf{e}^3 \cdot \mathbf{e}^1 = 0, \qquad g^{32} = \mathbf{e}^3 \cdot \mathbf{e}^2 = 0, \qquad g^{33} = \mathbf{e}^3 \cdot \mathbf{e}^3 = 1. \tag{48c}$$

Verify that $g^{ij} g_{jk} = \delta^i_k$.

(e) Completeness relation: Verify the completeness relation

$$\mathbf{e}^i \mathbf{e}_i = \mathbf{1} \tag{49}$$

by evaluating

$$\mathbf{e}^1 \mathbf{e}_1 + \mathbf{e}^2 \mathbf{e}_2 + \mathbf{e}^3 \mathbf{e}_3. \tag{50}$$

(f) Given a vector

$$\mathbf{A} = a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}} \tag{51}$$

in rectangular coordinates, find the components of the vector $\mathbf{A}$ in the basis of $\mathbf{e}_i$. That is, find the components A^i in

$$\mathbf{A} = A^1\,\mathbf{e}_1 + A^2\,\mathbf{e}_2 + A^3\,\mathbf{e}_3. \tag{52}$$