Homework No. 04 (Fall 2025)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2025 Sep 15, 4.30pm

1. (**20 points.**) For a given complex number z, say
$$z = \sqrt{2}\, e^{i\frac{\pi}{3}}, \tag{1}$$
evaluate
$$z^2, z^3, z^4, z^5, z^6, z^7, z^8, z^9, z^{10}. \tag{2}$$
Mark all of them on the complex plane. Decipher the pattern.

2. (**20 points.**) Evaluate
$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{23}. \tag{3}$$
Mark the resulting number on the complex plane.

3. (**20 points.**) Find the fifth roots of unity by solving the equation
$$z^5 = 1. \tag{4}$$
Mark the points corresponding to the five roots on the complex plane. Find the five roots of the equation
$$z^5 = -1. \tag{5}$$
Mark the roots on the complex plane. Next, find the roots of the equation
$$z^5 = i \tag{6}$$
and mark the roots on the complex plane. Repeat the exercise for $z^5 = -i$. How do these roots match with the fifth roots of unity? Recognize the pattern.

4. (**20 points.**) Recall that analytic functions satisfy the Cauchy-Riemann equations. That is, the real and imaginary parts of an analytic function
$$f(x + iy) = u(x, y) + iv(x, y) \tag{7}$$
satisfy
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{8a}$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{8b}$$

Given $f(z)$ and $g(z)$ are analytic functions in a region, then show that $f(g(z))$ satisfies the Cauchy-Riemann equations there.

Hint: Let $g = u + iv$ and $f = U + iV$. Thus, we can write

$$f(g(z)) = U(u(x,y), v(x,y)) + iV(u(x,y), v(x,y)). \tag{9}$$

5. (**20 points.**) Analytic functions are significantly constrained, in that they have to satisfy the Cauchy-Riemann conditions. These conditions are necessary (but not sufficient) for a function of a complex variable to be analytic (differentiable). Check if the following functions satisfy the Cauchy-Riemann conditions. If $f(z)$ is analytic for all z, then report the derivative as a function of z. Otherwise, determine the points, or regions, in the z plane where the function is not analytic.

$$f(z) = z^3, \tag{10a}$$
$$f(z) = |z|^2, \tag{10b}$$
$$f(z) = e^{iz}, \tag{10c}$$
$$f(z) = \ln z, \tag{10d}$$
$$f(z) = e^z + e^{iz}. \tag{10e}$$

Mathematica allows visualization of these functions using the command ComplexContourPlot.

```
f[z_] = z^3;
ComplexContourPlot[ReIm[f[z]], {z,-3-3 I,3+3 I}]
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The above two-line code in Mathematica plots the real and imaginary surfaces associated with the analytic function $f(z) = z^3$ between the coordinate points $(-3,-3)$ and $(3,3)$.

Use ComplexContourPlot in Mathematica to visualize these functions.