# Homework No. 06 (Fall 2025)

## PHYS 500A: MATHEMATICAL METHODS

*School of Physics and Applied Physics, Southern Illinois University–Carbondale*

Due date: Monday, 2025 Oct 6, 4.30pm

1. (**20 points.**) The Pauli matrices satisfy

$$\sigma_i \sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k. \tag{1}$$

A particular representation of Pauli matrices is

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

In particular, these are Pauli matrices in the eigenbasis of $\sigma_z$. Construct the matrix

$$\sigma_{\theta,\phi} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}(\theta, \phi) = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}, \tag{3}$$

where

$$\boldsymbol{\sigma} = \sigma_x \hat{\mathbf{i}} + \sigma_y \hat{\mathbf{j}} + \sigma_z \hat{\mathbf{k}}, \tag{4}$$

$$\hat{\mathbf{n}}(\theta, \phi) = \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}}. \tag{5}$$

Find the eigenvalues $\sigma'_{\theta,\phi}$ and the normalized eigenvectors, $|\sigma'_{\theta,\phi} = +1\rangle$ and $|\sigma'_{\theta,\phi} = -1\rangle$, (up to a phase) of the matrix $\sigma_{\theta,\phi}$.

2. (**20 points.**) Let

$$\sigma(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{6}$$

   (a) Define operator $a = \sigma(0)$. Find the eigenvalues $a'_i$ and eigenvectors $|a'_i\rangle$ of the operator $a$. Verify the completeness relation

$$1 = |a'_1\rangle\langle a'_1| + |a'_2\rangle\langle a'_2|. \tag{7}$$

   (b) Define operator $b = \sigma(\pi/2)$. Find the eigenvalues $b'_i$ and eigenvectors $|b'_i\rangle$ of the operator $b$. Verify the completeness relation

$$1 = |b'_1\rangle\langle b'_1| + |b'_2\rangle\langle b'_2|. \tag{8}$$

(c) The probability of finding the eigenstate $|b'_f\rangle\langle b'_f|$ in a measurement of $b$, after selecting $|a'_i\rangle\langle a'_i|$ in a measurement of $a$, is

$$p\big(|a'_i\rangle\langle a'_i| \to |b'_f\rangle\langle b'_f|\big) = \mathrm{tr}\big(|a'_i\rangle\langle a'_i|b'_f\rangle\langle b'_f|\big). \tag{9}$$

Evaluate the following probabilities.

$$p\big(|a'_1\rangle\langle a'_1| \to |b'_1\rangle\langle b'_1|\big) = \tag{10a}$$
$$p\big(|a'_1\rangle\langle a'_1| \to |b'_2\rangle\langle b'_2|\big) = \tag{10b}$$
$$p\big(|a'_2\rangle\langle a'_2| \to |b'_1\rangle\langle b'_1|\big) = \tag{10c}$$
$$p\big(|a'_2\rangle\langle a'_2| \to |b'_2\rangle\langle b'_2|\big) = \tag{10d}$$

(d) The probabilities for measurements and the associated selections involving more than two physical acts in a sequential order is given by

$$p\big(|a'_i\rangle\langle a'_i| \to |b'_j\rangle\langle b'_j| \to |a'_k\rangle\langle a'_k|\big) = \Big|\mathrm{tr}\big(\langle a'_i|b'_j\rangle\langle b'_j|a'_k\rangle\big)\Big|^2. \tag{11}$$

Observe that reading out probability amplitudes involves amputing the ket and bra of the initial and final state, respectively. Evaluate

$$p\big(|a'_1\rangle\langle a'_1| \to |b'_1\rangle\langle b'_1| \to |a'_1\rangle\langle a'_1|\big) = \tag{12}$$

and describe under what circumstances does a measurement disturb the system.