Homework No. 07 (Fall 2025)

PHYS 500A: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Friday, 2025 Oct 17, 4.30pm

- (**Notation.**) The Fourier space is spanned by the Fourier eigenfunctions

$$e^{im\phi}, \qquad m = 0, \pm 1, \pm 2, \ldots, \qquad 0 \le \phi < 2\pi. \tag{1}$$

An arbitrary function $f(\phi)$ has the Fourier series representation

$$f(\phi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} a_m e^{im\phi}, \tag{2}$$

where $e^{im\phi}$ are the Fourier eigenfunctions and a_m are the respective Fourier components.

1. (**20 points.**) Determine all the Fourier components a_m for the following functions: $\cos\phi$, $\sin\phi$, $\cos^2\phi$, $\sin^2\phi$, $\cos^3\phi$, $\sin^3\phi$.

2. (**20 points.**) Determine the particular function $f(\phi)$ that has the Fourier components

$$a_m = 1 \tag{3}$$

for all m. That is, all the Fourier coefficients are contributing equally in the series.

3. (**20 points.**) To determine the Fourier components of $\tan\phi$ start from

$$\tan\phi = \frac{1}{i} \frac{e^{i\phi} - e^{-i\phi}}{e^{i\phi} + e^{-i\phi}} \tag{4}$$

and show that

$$\tan\phi = \frac{1}{i} + \sum_{m=1}^{\infty} e^{-2im\phi} \frac{2(-1)^m}{i}. \tag{5}$$

Thus, read out all the Fourier components. Similarly, find the Fourier components of $\cot\phi$.

- (**Notation.**) The (continuous) Fourier space is spanned by the Fourier eigenfunctions

$$e^{ikx}, \qquad -\infty < k < \infty, \qquad -\infty < x < \infty. \tag{6}$$

An arbitrary function $f(x)$ has the Fourier series representation

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k), \tag{7}$$

where e^{ikx} are the Fourier eigenfunctions and $\tilde{f}(k)$ are the respective Fourier components.

4. (**20 points.**) Find the Fourier transform of a Gaussian function

$$f(x) = e^{-ax^2}. \tag{8}$$

That is, evalaute the integral

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx \, e^{-ikx} e^{-ax^2}. \tag{9}$$

5. (**20 points.**) The Heaviside step function is defined as

$$\theta(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases} \tag{10}$$

The Fourier transform and the corresponding inverse ae,

$$\theta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{-i\omega t} \tilde{\theta}(\omega), \tag{11a}$$

$$\tilde{\theta}(\omega) = \int_{-\infty}^{\infty} dt \, e^{i\omega t} \theta(t). \tag{11b}$$

(a) Using the definition in Eq. (10) in Eq. (11b) show that

$$\tilde{\theta}(\omega) = \int_0^{\infty} dt \, e^{i\omega t} = \lim_{\delta \to 0+} \int_0^{\infty} dt \, e^{i\omega t} e^{-\delta t} = \lim_{\delta \to 0+} -\frac{1}{i} \frac{1}{\omega + i\delta}. \tag{12}$$

(b) Verify that

$$\theta(t) = \lim_{\delta \to 0+} -\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega t}}{\omega + i\delta} \tag{13}$$

is indeed an integral representation of Heaviside step function.

6. (**20 points.**) Consider the inhomogeneous linear differential equation

$$\left(a \frac{d^2}{dx^2} + b \frac{d}{dx} + c \right) f(x) = \delta(x). \tag{14}$$

Use the Fourier transformation and the associated inverse Fourier transformation

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k), \tag{15a}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x), \tag{15b}$$

to show that the corresponding equation satisfied by $\tilde{f}(k)$ is algebraic. Find $\tilde{f}(k)$.

7. (**20 points.**) [Optional, Quantum Fourier Transform] An eigenbasis that spans an n-dimensional space consists of eigenvectors $\hat{\mathbf{e}}_i$, where $i = 1, 2, \ldots, n$. These eigenvectors have n components that can be indexed using $a, b = 1, 2, \ldots, n$. That is, $\hat{\mathbf{e}}_i = \mathrm{e}_i^a$. Thus, using Einstein summation convention, the orthonormality conditions can be stated as

$$\hat{\mathbf{e}}_i^\dagger \cdot \hat{\mathbf{e}}_j = \delta_{ij}, \qquad \text{or} \qquad \mathrm{e}_i^{a\dagger} \mathrm{e}_j^a = \delta_{ij}, \tag{16}$$

and the completeness relation can be stated as

$$\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1^\dagger + \ldots + \hat{\mathbf{e}}_n \hat{\mathbf{e}}_n^\dagger = \mathbf{1}, \qquad \text{or} \qquad \mathrm{e}_i^a \mathrm{e}_i^{b\dagger} = \delta^{ab}. \tag{17}$$

In this spirit, consider the following eigenbasis, constructed using n-th roots of unity,

$$\mathrm{e}_k^l = \frac{1}{\sqrt{n}} (u_k')^l = \frac{1}{\sqrt{n}} e^{i \frac{2\pi}{n} kl}, \qquad k, l = 1, 2, \ldots, n. \tag{18}$$

Show that

$$\sum_{k=1}^{n} e^{i \frac{2\pi}{n} k(l-l')} = n\, \delta_{ll'}, \tag{19}$$

and using this relation verify that the eigenbasis satisfies the completeness and orthonormality relations. For $n = 2$, the eigenvectors are

$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \hat{\mathbf{e}}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{20}$$

Show that these eigenvectors satisfy the orthonormality and completeness relations. Determine the eigenvectors for $n = 3$ and verify the corresponding completeness and orthonormality relations. Caution: Do not forget the complex conjugation.