Notes on Mathematical Physics

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1. These are notes prepared for the benefit of students enrolled in Mathematical Methods in Physics (PHYS-500A) at Southern Illinois University–Carbondale. It will be updated periodically, and will evolve during the semester. It is not a substitute for standard textbooks, but a supplement prepared as a study-guide.

2. The following textbooks were extensively used in this compilation.

   (a) Classical Electrodynamics,
   Julian Schwinger, Lester L. Deraad Jr., Kimball A. Milton, and Wu-yang Tsai,
   Advanced Book Program (1998)

   (b) Introduction to Classical and Modern Analysis and Their Application to Group Representation Theory,
   Debabrata Basu,
   World Scientific Publishing Co. Pte. Ltd. (2011)

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Contents

Chapter 1

Mathematical preliminaries

1.1 Vector calculus

1.1.1 (Algebraic) index notation

1. (**10 points.**) Verify the following relations:

$$\delta_{ij} = \delta_{ji}, \tag{1.1a}$$
$$\delta_{ii} = 3, \tag{1.1b}$$
$$\delta_{ik}\delta_{kj} = \delta_{ij}, \tag{1.1c}$$
$$\delta_{im}B_m = B_i, \tag{1.1d}$$
$$\varepsilon_{ijk} = -\varepsilon_{ikj} = \varepsilon_{kij}, \tag{1.1e}$$
$$\varepsilon_{iik} = 0, \tag{1.1f}$$
$$\delta_{ij}\varepsilon_{ijk} = 0. \tag{1.1g}$$

2. (**10 points.**) In three dimensions the Levi-Civita symbol is given in terms of the determinant of the Kronecker δ-functions,

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{lmn} &= \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \\ &= \delta_{il}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) \\ &\quad +\delta_{im}(\delta_{jn}\delta_{kl} - \delta_{jl}\delta_{kn}) \\ &\quad +\delta_{in}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}).\end{aligned} \tag{1.2a}$$

Using the above identity show that

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}, \tag{1.3a}$$
$$\varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn}, \tag{1.3b}$$
$$\varepsilon_{ijk}\varepsilon_{ijk} = 6. \tag{1.3c}$$

3. (**10 points.**) Using the property of Kronecker δ-function and Levi-Civita symbol evaluate the following using index notation.

$$\delta_{ij}\delta_{ji} = \tag{1.4a}$$
$$\delta_{ij}\varepsilon_{ijk} = \tag{1.4b}$$
$$\varepsilon_{ijm}\delta_{mn}\varepsilon_{nij} = \tag{1.4c}$$

4. (**20 points.**) Using index notation and the properties of Kronecker δ-function and Levi-Civita symbol expand the left hand side of the vector equation below to express it in the form on the right hand side,
$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = \alpha(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) + \beta(\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}). \tag{1.5}$$
In particular find the numbers α and β.

5. (**20 points.**) Given
$$\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r} \tag{1.6}$$
where $\mathbf{B}$ is a constant (homogeneous in space) vector field. Using index notation and the properties of Kronecker δ-function and Levi-Civita symbol in three dimensions expand the left hand side of the vector equation below to express it in the form on the right hand side,
$$\boldsymbol{\nabla}\times\mathbf{A} = \alpha\mathbf{B} + \beta\mathbf{r}. \tag{1.7}$$
In particular find the numbers α and β.

6. (**10 points.**) Derive the following vector identities (using index notation)
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}), \tag{1.8}$$
$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}), \tag{1.9}$$

7. (**20 points.**) For a vector field $\mathbf{A}$, evaluate the vector identity
$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}). \tag{1.10}$$
Then, after the introduction of a scalar field ψ, evaluate
$$\boldsymbol{\nabla}\left[\psi\cdot(\boldsymbol{\nabla}\times\mathbf{A})\right]. \tag{1.11}$$

[2023F-MT01]

8. (**10 points.**) Use index notation or dyadic notation to show that
$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}, \tag{1.12a}$$
$$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{B} - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B}), \tag{1.12b}$$
$$\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} + \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - (\boldsymbol{\nabla}\cdot\mathbf{A})\mathbf{B} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}. \tag{1.12c}$$

9. (**10 points.**) (Ref. Schwinger et al., problem 1, chapter 1.) Verify the following identities explicitly:
   (a) $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}$,
   (b) $\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = (\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{B} - \mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})$,
   (c) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = 0$,
   (d) $\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) - \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) - (\mathbf{A}\times\boldsymbol{\nabla})\times\mathbf{B} + (\mathbf{B}\times\boldsymbol{\nabla})\times\mathbf{A}$.

1.1.2 (Geometric) dyadic notation

1. (**20 points.**) Verify the following identities:
$$\boldsymbol{\nabla} r = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}, \tag{1.13a}$$
$$\boldsymbol{\nabla}\,\mathbf{r} = \mathbf{1}. \tag{1.13b}$$

Further, show that

$$\nabla \cdot \mathbf{r} = 3, \tag{1.14a}$$
$$\nabla \times \mathbf{r} = 0. \tag{1.14b}$$

Here r is the magnitude of the position vector $\mathbf{r}$, and $\hat{\mathbf{r}}$ is the unit vector pointing in the direction of $\mathbf{r}$.

2. (**25 points.**) Evaluate

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right), \tag{1.15}$$

everywhere in space, including $\mathbf{r} = 0$.
Hint: Check your answer for consistency by using divergence theorem.

3. (**10 points.**) Show that

   (a) $\nabla \frac{1}{r^n} = -\mathbf{r}\, \frac{n}{r^{n+2}}$

   (b) $\nabla \frac{\mathbf{r}}{r^n} = \mathbf{1}\, \frac{1}{r^n} - \mathbf{r}\,\mathbf{r}\, \frac{n}{r^{n+2}}$

   (c) $\nabla \cdot \frac{\mathbf{r}}{r^n} = \frac{(3-n)}{r^n}$

   (d) $\nabla \times \frac{\mathbf{r}}{r^n} = 0$

4. (**10 points.**) For the position vector

$$\mathbf{r} = r\,\hat{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}, \tag{1.16}$$

show that

$$\nabla r = \hat{\mathbf{r}}, \quad \nabla\,\mathbf{r} = \mathbf{1}, \quad \nabla \cdot \mathbf{r} = 3, \quad \text{and} \quad \nabla \times \mathbf{r} = 0. \tag{1.17}$$

Further, show that for $n \neq 3$

$$\nabla \frac{\mathbf{r}}{r^n} = \mathbf{1}\frac{1}{r^n} - \mathbf{r}\,\mathbf{r}\frac{n}{r^{n+2}}, \quad \nabla \cdot \frac{\mathbf{r}}{r^n} = \frac{(3-n)}{r^n}, \quad \text{and} \quad \nabla \times \frac{\mathbf{r}}{r^n} = 0. \tag{1.18}$$

For $n = 3$ use divergence theorem to show that

$$\nabla \cdot \frac{\mathbf{r}}{r^n} = 4\pi\, \delta^{(3)}(\mathbf{x}). \tag{1.19}$$

5. (**10 points.**) (Based on Problem 1.13, Griffiths 4th edition.)
Show that

$$\nabla r^2 = 2\mathbf{r}. \tag{1.20}$$

Then evaluate ∇r^3. Show that

$$\nabla \frac{1}{r} = -\frac{\hat{\mathbf{r}}}{r^2}. \tag{1.21}$$

Then evaluate $\nabla(1/r^2)$.

6. (**10 points.**) Evaluate the left hand side of the equation

$$\nabla \frac{1}{r^3} = \alpha\, \hat{\mathbf{r}}\, r^n. \tag{1.22}$$

Thus find α and n.

7. (**10 points.**) Evaluate the left hand side of the equation
$$\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{p}) = a\,\mathbf{p} + b\,\mathbf{r}, \tag{1.23}$$
where $\mathbf{p}$ is a constant vector. Thus, find a and b.

8. (**20 points.**) Evaluate
$$\nabla^2\left(\frac{1}{\mathbf{a}\cdot\mathbf{r}}\right), \tag{1.24}$$
where $\mathbf{a}$ is a constant vector.

9. (**20 points.**) Given
$$\nabla^2(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r}) = c. \tag{1.25}$$
Find the scalar c.

10. (**20 points.**) Evaluate the left hand side of the equation
$$\boldsymbol{\nabla}\cdot(r^2\mathbf{r}) = a\,r^n. \tag{1.26}$$
Thus, find a and n.

11. (**20 points.**) Evaluate
$$\boldsymbol{\nabla}\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right), \tag{1.27}$$
where $\mathbf{p}$ is a constant vector.

12. (**20 points.**) Evaluate the left hand side of the equation
$$\boldsymbol{\nabla}\left(\frac{1}{\mathbf{r}\cdot\mathbf{p}}\right) = a\,\mathbf{p} + b\,\mathbf{r}, \tag{1.28}$$
where $\mathbf{p}$ is a constant vector. Thus, find a and b.

13. (**20 points.**) Evaluate
$$\boldsymbol{\nabla}\times\left(\frac{\mathbf{m}\times\mathbf{r}}{r^3}\right), \tag{1.29}$$
where $\mathbf{m}$ is a constant vector.

14. (**20 points.**) Given the flow velocity field
$$\mathbf{v} = \omega\rho\hat{\boldsymbol{\phi}} \tag{1.30}$$
determine the vorticity $\boldsymbol{\nabla}\times\mathbf{v}$ of the flow. Illustrate the flow field and the vorticity using the associated vector field lines. Here ω is a constant, and ρ and ϕ are cylindrical polar coordinates.

15. (**20 points.**) Given the flow velocity field
$$\mathbf{v} = \frac{c}{\rho}\hat{\boldsymbol{\phi}} \tag{1.31}$$
determine the vorticity $\boldsymbol{\nabla}\times\mathbf{v}$ of the flow. Illustrate the flow field and the vorticity using the associated vector field lines. Here c is a constant, and ρ and ϕ are cylindrical polar coordinates. Let $\rho \neq 0$.

16. (**20 points.**) The relation between the vector potential $\mathbf{A}$ and the magnetic field $\mathbf{B}$ is
$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{1.32}$$
For a constant (homogeneous in space) magnetic field $\mathbf{B}$, verify that
$$\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r} \tag{1.33}$$
is a possible vector potential by showing that Eq. (1.33) satisfies Eq. (1.32).

17. (**20 points.**) Evaluate

$$(\mathbf{a} \times \boldsymbol{\nabla}) \cdot (\mathbf{r} \times \mathbf{b}), \tag{1.34}$$

where $\mathbf{r}$ is the the coordinate vector and $\mathbf{a}$ and $\mathbf{b}$ are coordinate independent vectors. [2023F-MT01]

18. (**20 points.**) Show that

$$\boldsymbol{\nabla}(\hat{\mathbf{r}} \cdot \mathbf{a}) = -\frac{1}{r}\,\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{a}) \tag{1.35}$$

for a uniform (homogeneous in space) vector $\mathbf{a}$.

19. (**25 points.**) Show that

$$\boldsymbol{\nabla} \cdot \big[P_0\,\hat{\mathbf{r}}\,\theta(R-r)\big], \tag{1.36}$$

for a uniform (homogeneous in space) P_0, can be expressed as a sum of two terms, a surface term and a volume term. Here $\theta(x) = 1$ if $x > 0$ and 0 otherwise.

20. (**20 points.**) Consider the dyadic construction of an unitary operator

$$\mathbf{U} = \hat{\mathbf{i}}\,\hat{\mathbf{j}} + \hat{\mathbf{j}}\,\hat{\mathbf{i}}, \tag{1.37}$$

where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are orthonormal vectors satisfying the completeness relation

$$\mathbf{1} = \hat{\mathbf{i}}\,\hat{\mathbf{i}} + \hat{\mathbf{j}}\,\hat{\mathbf{j}}. \tag{1.38}$$

Evaluate

$$\mathrm{tr}(\mathbf{U}^{107}). \tag{1.39}$$

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21. (**20 points.**) Consider the dyadic construction

$$\mathbf{T} = \mathbf{E}\,\mathbf{B} \tag{1.40}$$

built using the vector fields,

$$\mathbf{E} = E_x\,\hat{\mathbf{x}} + E_y\,\hat{\mathbf{y}}, \tag{1.41a}$$

$$\mathbf{B} = B\,\hat{\mathbf{y}}. \tag{1.41b}$$

Evaluate the following components of the dyadic:

$$\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} = \tag{1.42a}$$

$$\hat{\mathbf{y}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \hat{\mathbf{y}} \cdot \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \hat{\mathbf{y}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} = \tag{1.42b}$$

$$\hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} = \tag{1.42c}$$

Evaluate the scalars

$$\mathrm{Tr}(\mathbf{T}) = T_{ii}, \tag{1.43a}$$

$$\mathrm{Tr}(\mathbf{T} \cdot \mathbf{T}) = T_{ij}T_{ji}, \tag{1.43b}$$

$$\mathrm{Tr}(\mathbf{T} \cdot \mathbf{T} \cdot \mathbf{T}) = T_{ij}T_{jk}T_{ki}. \tag{1.43c}$$

Evaluate the following vector field constructions:

$$\hat{\mathbf{x}} \cdot \mathbf{T} = \qquad \hat{\mathbf{y}} \cdot \mathbf{T} = \qquad \hat{\mathbf{z}} \cdot \mathbf{T} = \tag{1.44a}$$

$$\mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \mathbf{T} \cdot \hat{\mathbf{z}} = \tag{1.44b}$$

$$\hat{\mathbf{x}} \times \mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \hat{\mathbf{y}} \times \mathbf{T} \cdot \hat{\mathbf{x}} = \qquad \hat{\mathbf{z}} \times \mathbf{T} \cdot \hat{\mathbf{x}} = \tag{1.44c}$$

$$\hat{\mathbf{x}} \times \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \hat{\mathbf{y}} \times \mathbf{T} \cdot \hat{\mathbf{y}} = \qquad \hat{\mathbf{z}} \times \mathbf{T} \cdot \hat{\mathbf{y}} = \tag{1.44d}$$

$$\hat{\mathbf{x}} \times \mathbf{T} \cdot \hat{\mathbf{z}} = \qquad \hat{\mathbf{y}} \times \mathbf{T} \cdot \hat{\mathbf{z}} = \qquad \hat{\mathbf{z}} \times \mathbf{T} \cdot \hat{\mathbf{z}} = \tag{1.44e}$$

$$\hat{\mathbf{x}} \cdot \mathbf{T} \times \hat{\mathbf{x}} = \qquad \hat{\mathbf{x}} \cdot \mathbf{T} \times \hat{\mathbf{y}} = \qquad \hat{\mathbf{x}} \cdot \mathbf{T} \times \hat{\mathbf{z}} = \tag{1.44f}$$

$$\hat{\mathbf{y}} \cdot \mathbf{T} \times \hat{\mathbf{x}} = \qquad \hat{\mathbf{y}} \cdot \mathbf{T} \times \hat{\mathbf{y}} = \qquad \hat{\mathbf{y}} \cdot \mathbf{T} \times \hat{\mathbf{z}} = \tag{1.44g}$$

$$\hat{\mathbf{z}} \cdot \mathbf{T} \times \hat{\mathbf{x}} = \qquad \hat{\mathbf{z}} \cdot \mathbf{T} \times \hat{\mathbf{y}} = \qquad \hat{\mathbf{z}} \cdot \mathbf{T} \times \hat{\mathbf{z}} = \tag{1.44h}$$

1.2 Misellaneous

1.2.1 Vector differentiation

1. (**10 points.**) A gyroid, see Fig. 1.1, is an (infinitely connected triply periodic minimal) surface discovered by Alan Schoen in 1970. Schoen presently resides in Carbondale and was a professor at SIU in the later part of his career. Apparently, a gyroid is approximately described by the surface

$$f(x, y, z) = \cos x \sin y + \cos y \sin z + \cos z \sin x \tag{1.45}$$

when $f(x, y, z) = 0$. Using the fact that the gradient operator

$$\boldsymbol{\nabla} = \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \tag{1.46}$$

determines the normal vectors on a surface, evaluate

$$\boldsymbol{\nabla} f(x, y, z). \tag{1.47}$$

2. (**30 points.**) (Based on problem 1.26 Griffiths 4th edition.)
Calculate the Laplacian of the following functions:

   (a) $T_a = x^2 + 2xy + 3z + 4$
   (b) $T_b = \sin x \sin y \sin z$
   (c) $\mathbf{v} = x^2 \, \hat{\mathbf{x}} + 3xz^2 \, \hat{\mathbf{y}} - 2xz \, \hat{\mathbf{z}}$

1.2.2 Vector integration

1. (**10 points.**) (Based on problem 1.32/1.31 Griffiths 4th/3rd edition.)
Check the fundamental theorem for gradients,

$$\int_{\mathbf{a}}^{\mathbf{b}} d\mathbf{l} \cdot \boldsymbol{\nabla} T = T(\mathbf{b}) - T(\mathbf{a}), \tag{1.48}$$

using $T = x^2 + 4xy + 2yz^3$, the points $\mathbf{a} = (0, 0, 0)$, $\mathbf{b} = (1, 1, 1)$, and the three paths of Fig. 1.28 in Griffiths.

2. (**10 points.**) (Based on problem 1.33/1.32 Griffiths 4th/3rd edition.)
Check the fundamental theorem of divergence,

$$\int_V d^3x \, \boldsymbol{\nabla} \cdot \mathbf{E} = \oint_S d\mathbf{a} \cdot \mathbf{E}, \tag{1.49}$$

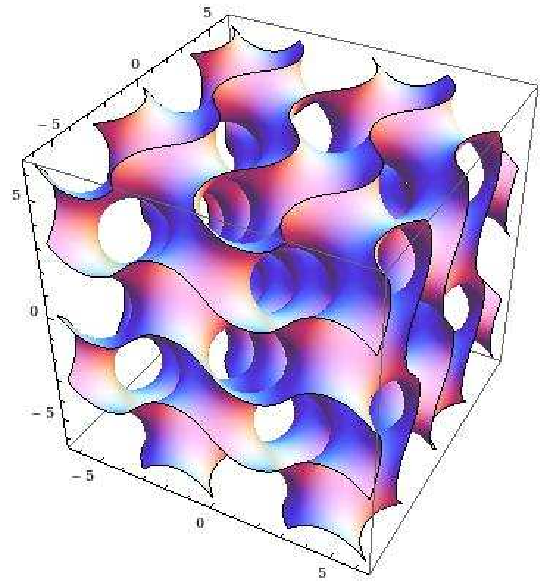

Figure 1.1: Problem 1.

for the vector field $\mathbf{E} = x\,\hat{\mathbf{x}}$. Use the volume V to be a cube of length L with an edge of the cube parallel to the x-axis. Using the fact that the divergence of a vector field at a point tells us whether a point is a source or sink of the field, estimate the distribution of the source and sink for the field $\mathbf{E}$?

3. (**20 points.**) Evaluate the flux,

$$\int_S d\mathbf{a} \cdot \mathbf{E}, \tag{1.50}$$

of the uniform (homogeneous in space) field

$$\mathbf{E} = E\,\hat{\mathbf{z}} \tag{1.51}$$

through the following surfaces:

   (a) A hemispherical bowl of radius R placed such that the circle determining the edge of the hemisphere is on the x-y plane. Show that the result is independent of the position of the center of the circle.

   (b) A semicircular cylinder of radius R and length L placed on the x-y plane.

4. (**10 points.**) (Based on problem 1.34/1.33 Griffiths 4th/3rd edition.)
Check the fundamental theorem of curl,

$$\int_S d\mathbf{a} \cdot \boldsymbol{\nabla} \times \mathbf{E} = \oint_C d\mathbf{l} \cdot \mathbf{E}, \tag{1.52}$$

(where the sense of the line integration is given by the right hand rule: the contour C is traversed in the sense of the fingers of the right hand and the thumb points in the sense of the orientation of the surface,) for the vector field $\mathbf{E} = y\,\hat{\mathbf{x}} + z\,\hat{\mathbf{y}} + x\,\hat{\mathbf{z}}$. Use the surface S to be a square of length L on the $z = 0$ plane with one side parallel to the x-axis. Using the fact that the curl of a vector field at a point is a measure of the torque experienced by a (point) dipole at the point, estimate the torque field.

5. (**20 points.**) Evaluate the vector area of a hemispherical bowl of radius R given by

$$\mathbf{a} = \int_S d\mathbf{a}, \tag{1.53}$$

where S stands for the surface of the hemispherical bowl. Next, evaluate the above vector area on the surface of a sphere.

6. (**20 points.**) Evaluate the vector area of a spherical ball of radius R using

$$\mathbf{a} = \int_S d\mathbf{a}, \tag{1.54}$$

where S stands for the surface of the spherical ball. (Caution: The question is discussing the vector area, which is different from the typical surface area of a sphere.)

1.2.3 Curvilinear coordinates

1. (**10 points.**) In spherical polar coordinates a point is coordinated by the intersection of family of spheres, cones, and half-planes, given by

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{1.55a}$$

$$\theta = \tan^{-1}\sqrt{\frac{x^2+y^2}{z^2}}, \tag{1.55b}$$

$$\phi = \tan^{-1}\frac{y}{x}, \tag{1.55c}$$

respectively. Show that the gradient of these surfaces are given by

$$\boldsymbol{\nabla} r = \hat{\mathbf{r}}, \qquad \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}, \tag{1.56a}$$

$$\boldsymbol{\nabla}\theta = \hat{\boldsymbol{\theta}}\,\frac{1}{r}, \qquad \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}, \tag{1.56b}$$

$$\boldsymbol{\nabla}\phi = \hat{\boldsymbol{\phi}}\,\frac{1}{r\sin\theta}, \qquad \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}, \tag{1.56c}$$

which are normal to the respective surfaces. Sketch the surfaces and the corresponding normal vectors. This illustrates that $\boldsymbol{\nabla}$(surface) is a vector (field) normal to the surface.

2. (**20 points.**) Verify that, $\boldsymbol{\nabla}\mathbf{r} = \mathbf{1}$,

$$\hat{\mathbf{i}}\hat{\mathbf{i}} + \hat{\mathbf{j}}\hat{\mathbf{j}} + \hat{\mathbf{k}}\hat{\mathbf{k}} = \mathbf{1}, \tag{1.57a}$$

$$\hat{\mathbf{r}}\hat{\mathbf{r}} + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}} = \mathbf{1}. \tag{1.57b}$$

3. (**20 points.**) Show that

$$\frac{\partial}{\partial\phi}\hat{\boldsymbol{\phi}} = -\big[\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}\big], \tag{1.58}$$

where (r,θ,ϕ) are spherical coordinates and $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ are the respective unit vectors in spherical coordinates. Sketch $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$, and $\partial\hat{\boldsymbol{\phi}}/\partial\phi$ to illustrate their relative directions.

4. (**20 points.**) Evaluate the number evaluated by the expression

$$\frac{1}{2}\left[\hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi}\right]\cdot(\rho\hat{\boldsymbol{\rho}}), \tag{1.59}$$

where $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ are the unit vectors for cylindrical coordinates (ρ,ϕ) given by

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}, \tag{1.60}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}. \tag{1.61}$$

5. (**25 points.**) Evaluate the number evaluated by the expression

$$\hat{\boldsymbol{\phi}} \cdot \left(\frac{1}{\rho} \frac{\partial}{\partial \phi} \rho \right) \hat{\boldsymbol{\rho}}, \tag{1.62}$$

where $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ are the unit vectors for cylindrical coordinates (ρ, ϕ) given by

$$\hat{\boldsymbol{\rho}} = \cos\phi\, \hat{\mathbf{i}} + \sin\phi\, \hat{\mathbf{j}}, \tag{1.63}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\, \hat{\mathbf{i}} + \cos\phi\, \hat{\mathbf{j}}. \tag{1.64}$$

6. (**20 points.**) The gradient operator in cylinderical coordinates (ρ, ϕ, z) is

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}} \frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{\mathbf{z}} \frac{\partial}{\partial z}. \tag{1.65}$$

The electric field of an infinitely long rod of negligible thickness is given by

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{2\lambda}{\rho} \hat{\boldsymbol{\rho}}, \tag{1.66}$$

where λ is the charge per unit length on the rod. Evaluate

$$\boldsymbol{\nabla} \cdot \mathbf{E}. \tag{1.67}$$

Hint: The divergence of electric field at a point in space is a measure of the charge density at that point. It satisfies the Gauss's law.

7. (**10 points.**) Determine the right hand side of the following expression for all $\mathbf{r}$. (You do not need to show your work.)

$$\boldsymbol{\nabla} \cdot \frac{\mathbf{r}}{r^3} = \tag{1.68}$$

1.2.4 Divergence in curvilinear coordinates

1. (**20 points.**) Evaluate

$$\boldsymbol{\nabla} \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right), \tag{1.69}$$

where $\mathbf{r}$ is the coordinate vector. Deduce that Eq. (1.69) can not be zero everywhere in space. Express Eq. (1.69) in terms of δ-functions.

1.2.5 Curl in curvilinear coordinates

Let the unit vectors associated with curvilinear coordinates (ξ_1, ξ_2, ξ_3) be $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ and let (h_1, h_2, h_3) be the respective scale factors. The gradient operator in these coordinates has the form

$$\boldsymbol{\nabla} = \hat{\mathbf{e}}_1 \frac{1}{h_1} \frac{\partial}{\partial \xi_1} + \hat{\mathbf{e}}_2 \frac{1}{h_2} \frac{\partial}{\partial \xi_2} + \hat{\mathbf{e}}_3 \frac{1}{h_3} \frac{\partial}{\partial \xi_3}. \tag{1.70}$$

A vector field $\mathbf{N}(\xi_1, \xi_2, \xi_3)$ in these coordinates has the form

$$\mathbf{N} = N_1(\xi_1, \xi_2, \xi_3)\, \hat{\mathbf{e}}_1 + N_2(\xi_1, \xi_2, \xi_3)\, \hat{\mathbf{e}}_2 + N_3(\xi_1, \xi_2, \xi_3)\, \hat{\mathbf{e}}_3. \tag{1.71}$$

The curl in these coordinates can be evaluated as the determinant

$$\boldsymbol{\nabla} \times \mathbf{N} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial \xi_1} & \dfrac{\partial}{\partial \xi_2} & \dfrac{\partial}{\partial \xi_3} \\ h_1 N_1 & h_2 N_2 & h_3 N_3 \end{vmatrix}. \tag{1.72}$$

1. (**20 points.**) Given
$$\mathbf{N} = \hat{\mathbf{z}} \ln \rho \tag{1.73}$$
and the gradient operator
$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}} \frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \tag{1.74}$$
evaluate
$$\boldsymbol{\nabla} \times \mathbf{N}. \tag{1.75}$$

2. (**20 points.**) Given
$$\mathbf{N} = \frac{\hat{\boldsymbol{\phi}}}{\rho} \tag{1.76}$$
and the gradient operator
$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}} \frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \tag{1.77}$$
evaluate
$$\boldsymbol{\nabla} \times \mathbf{N}. \tag{1.78}$$
Here (ρ, ϕ, z) are cylindrical coordinate variables. Deduce that Eq. (1.78) can not be zero everywhere in space. Express Eq. (1.78) in terms of δ-functions.

3. (**20 points.**) Given
$$\mathbf{N} = \hat{\boldsymbol{\phi}} \frac{\rho}{(\rho^2 + z^2)^{\frac{3}{2}}} \tag{1.79}$$
and the gradient operator
$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}} \frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}} \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \tag{1.80}$$
evaluate
$$\boldsymbol{\nabla} \times \mathbf{N}. \tag{1.81}$$

4. (**20 points.**) Given
$$\mathbf{N} = \hat{\boldsymbol{\phi}} \frac{\sin \theta}{r^2} \tag{1.82}$$
and the gradient operator
$$\boldsymbol{\nabla} = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \tag{1.83}$$
evaluate
$$\boldsymbol{\nabla} \times \mathbf{N}. \tag{1.84}$$

1.3 Heaviside step-function

1. (**20 points.**) The Heaviside step function, named after Oliver Heaviside (1850-1925), has the integral representation
$$\theta(x) = \int_{-\infty}^{x} dx' \delta(x'). \tag{1.85}$$
   (a) Evaluate $\theta(x)$ for $x < 0$.
   (b) Evaluate $\theta(x)$ for $x > 0$.
   (c) What about $\theta(0)$? We could postulate that
$$\theta(0) = \frac{1}{2} \left[\lim_{\varepsilon \to 0} \theta(x - \varepsilon) + \lim_{\varepsilon \to 0} \theta(x + \varepsilon) \right]. \tag{1.86}$$
   Evaluate $\theta(0)$ obtained using Eq. (1.86).
   (d) Plot $\theta(x)$ versus x.

1.4 δ-function distributions

1. (**10 points.**) Assume $0 < a < b$. Consider the distribution

$$\delta(x-a) = \lim_{b\to a} \frac{\theta(\rho-a)\theta(b-\rho)}{(b-a)} \tag{1.87}$$

constructed using Heaviside step-functions. Show that

$$\delta(x-a) \begin{cases} \to \infty, & \text{if} \quad x = a, \\ \to 0, & \text{if} \quad x \neq a. \end{cases} \tag{1.88}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x-a) = 1. \tag{1.89}$$

Plot $\delta(x-a)$ before taking the limit $b \to a$ and identify the length $(b-a)$ in the plot.

2. (**10 points.**) Consider the distribution

$$\delta(x) = \lim_{\varepsilon\to 0} \frac{\varepsilon}{x^2+\varepsilon^2}\frac{1}{\pi}. \tag{1.90}$$

Show that

$$\delta(x) \begin{cases} \to \infty, & \text{if} \quad x = 0, \\ \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{1.91}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{1.92}$$

Plot $\delta(x)$ before taking the limit $\varepsilon \to 0$ and identify ε in the plot.

3. (**10 points.**) Consider the distribution

$$\delta(x) = \lim_{\epsilon\to 0} \frac{\epsilon^2}{(x^2+\epsilon^2)^{\frac{3}{2}}}\frac{1}{2}. \tag{1.93}$$

Show that

$$\delta(x) = \begin{cases} \to \frac{1}{\epsilon} \to \infty, & \text{if} \quad x = 0, \\ \to \frac{\epsilon^2}{x^3} \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{1.94}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{1.95}$$

Plot $\delta(x)$ before taking the limit $\varepsilon \to 0$ and identify ε in the plot.

4. (**10 points.**) Consider the distribution

$$\delta(x) = \lim_{\epsilon\to 0} \frac{\epsilon^{n-1}}{(x^2+\epsilon^2)^{\frac{n}{2}}}\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \tag{1.96}$$

Show that

$$\delta(x) = \begin{cases} \to \frac{1}{\epsilon} \to \infty, & \text{if} \quad x = 0, \\ \to \frac{\epsilon^{n-1}}{x^n} \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{1.97}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{1.98}$$

Plot $\delta(x)$ before taking the limit $\varepsilon \to 0$ and identify ε in the plot.

5. (**10 points.**) Consider the distribution

$$\delta(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}}. \tag{1.99}$$

Show that

$$\delta(x) \begin{cases} \to \infty, & \text{if} \quad x = 0, \\ \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{1.100}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{1.101}$$

Plot $\delta(x)$ before taking the limit $\sigma \to 0$ and identify σ in the plot.

6. (**10 points.**) Consider the distribution

$$\delta(x) = \lim_{\varepsilon\to 0} \frac{e^{-\frac{|x|}{\varepsilon}}}{2\varepsilon}. \tag{1.102}$$

Show that

$$\delta(x) \begin{cases} \to \infty, & \text{if} \quad x = 0, \\ \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{1.103}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{1.104}$$

Plot $\delta(x)$ before taking the limit $\varepsilon \to 0$ and identify ε in the plot.

7. (**10 points.**) Consider the distribution

$$\delta(x) = \lim_{N\to\infty} \int_{-Nk_0}^{Nk_0} \frac{dk}{2\pi}\, e^{ikx} = \lim_{N\to\infty} \frac{1}{\pi} \frac{\sin Nk_0 x}{x}. \tag{1.105}$$

Show that

$$\delta(x) \begin{cases} \to \infty, & \text{if} \quad x = 0, \\ \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{1.106}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{x} \sin x = 1. \tag{1.107}$$

Hint: The last integral is nicely evaluated by continuing to the complex plane. Otherwise, to evaluate the integral construct

$$I(a) = \int_0^{\infty} \frac{dx}{x}\, e^{-ax} \sin x \tag{1.108}$$

and learn that

$$-I'(a) = \int_0^{\infty} dx\, e^{-ax} \sin x. \tag{1.109}$$

Integrating by parts twice deduce

$$-I'(a) = \frac{1}{1+a^2} \tag{1.110}$$

and integrate to conclude

$$I(a) = \frac{\pi}{2} - \tan^{-1} a. \tag{1.111}$$

8. (**20 points.**) Justify the relation

$$\theta(x) = \int_{-\infty}^{x} dx'\, \delta(x'). \tag{1.112}$$

1.4.1 Application of δ-functions

1. (**70 points.**) (Based on problem 1.44,45/1.43,44 Griffiths 4th/3rd edition.) Evaluate the following integrals:

   (a) $\int_2^6 dx \left[3x^2 - 2x - 3\right] \delta(x-3)$

   (b) $\int_{-7}^{7} dx \sin x \, \delta(x - \pi)$

   (c) $\int_0^3 dx \, x^3 \, \delta(x+1)$

   (d) $\int_{-2}^{2} dx \, [3x+3] \, \delta(3x)$

   (e) $\int_{-2}^{2} dx \, [3x+3] \, \delta(-3x)$

   (f) $\int_0^2 dx \, [3x+3] \, \delta(1-x)$

   (g) $\int_{-1}^{1} dx \, 9x^3 \, \delta(3x+1)$

2. (**10 points.**) Evaluate the integral

$$\int_{-1}^{1} \frac{\delta(1-3x)}{x} dx. \tag{1.113}$$

   Hint: Be careful to avoid a possible error in sign.

3. (**20 points.**) Evaluate the integral

$$\int_{-1}^{1} dx \, \delta(1-2x) \Big[8x^2 + 2x - 1\Big]. \tag{1.114}$$

   (Caution: Be careful to avoid a possible error in sign.)

4. (**30 points.**) (Based on problem 1.47/1.46 Griffiths 4th/3rd edition.)

   (a) Express the charge density $\rho(\mathbf{r})$ of a point charge Q positioned at $\mathbf{r}_a$ in terms of δ-functions. Verify that the volume integral of ρ equals Q.

   (b) Express the charge density of an infinitely long wire, of uniform charge per unit length λ and parallel to z-axis, in terms of δ-functions.

   (c) Express the charge density of an infinite plate, of uniform charge per unit area σ and parallel to xy-plane, in terms of δ-functions.

5. Problem 1.2, Jackson 3rd edition.

6. Problem 1.3, Jackson 3rd edition.

7. (**10 points.**) An (idealized) infinitely long wire, (on the z-axis with infinitesimally small cross sectional area,) carrying a current I can be mathematically represented by the current density

$$\mathbf{J}(\mathbf{x}) = \hat{\mathbf{z}} \, I \, \delta(x)\delta(y). \tag{1.115}$$

   A similar idealized wire forms a circular loop and is placed on the xy-plane with the center of the circular loop at the origin. Write down the current density of the circular loop carrying current I.

8. (**10 points.**) A uniformly charged spherical shell of radius a and total charge Q is described by charge density

$$\rho(\mathbf{x}) = \frac{Q}{4\pi a^2} \delta(r-a). \tag{1.116}$$

   Verify that the volume integral of ρ equals Q.

9. (**10 points.**) A uniformly charged infinitely thin disc of radius a and total charge Q is placed on the $z = 0$ plane such that the normal vector on the disc is along the z axis and the center of the disc at the origin. Write down the charge density of the disc in terms of δ-function(s) and Heaviside step function(s). Integrate the charge density over all space to verify that it indeed returns the total charge on the disc. [2023F-MT01]

10. (**20 points.**) A uniformly charged infinitely thin wire of length a and total charge Q is placed on the z axis such that one end of the wire is at the origin. Write down the charge density of the wire in terms of δ-function(s) and Heaviside step function(s). Integrate the charge density over all space to verify that it indeed returns the total charge on the wire.

11. (**10 points.**) Write down the charge density for the following configurations: Point charge, line charge, surface charge, uniformly charged disc, uniformly charged ring, uniformly charged shell, uniformly charged spherical ball.

12. (**10 points.**) $\delta(ax)$, $\delta(ax+b)$ for $a>0$, $\delta(ax+b)$ for $a<0$, $\delta'(x)$, $\delta''(x)$,

13. (**10 points.**) The distance between two points $\mathbf{r}$ and $\mathbf{r}'$ in rectangular coordinates is explicitly given by

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}. \tag{1.117}$$

The charge density of a charge q at the origin is described in terms of delta functions as

$$\rho(\mathbf{r}) = q\delta(x)\delta(y)\delta(z). \tag{1.118}$$

Evaluate the electric potential at the observation point $\mathbf{r}$, due to a point charge q placed at source point $\mathbf{r}'$, using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{1.119}$$

where $\int d^3r' = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz'$. That is, evaluate the three integrals in

$$\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \frac{\delta(x')\delta(y')\delta(z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}. \tag{1.120}$$

14. (**10 points.**) Evaluate

$$\frac{d}{dz}|z| \tag{1.121}$$

and

$$\frac{d^2}{dz^2}|z|, \tag{1.122}$$

in terms of the Heaviside step function

$$\theta(z) = \begin{cases} 0, & z<0, \\ 1, & z>0, \end{cases} \tag{1.123}$$

and the delta function.

1.5 δ-functions in infinite dimensional spaces

δ-functions is the generalization of the unit matrix $\mathbf{1}$, in finite dimensional vector space, in infinite dimensional vector space. In particular, it is expected to contain the information regarding the dimensionality of the infinite dimensional. Recall $\mathrm{Tr}\,(\mathbf{1})$ is equal to the dimension of a finite dimensional vector space. However, recall that the dimension of a countable set in counter intuitive, but well defined.

Chapter 2

Vector space

2.1 Curvilinear coordinates

1. (**Example.**) Let $\mathbf{r}$ represent the position vector, x^i the components of the position vector in rectangular coordinates, and u^i the components of the position vector in cylindrical polar coordinates. In particular, we have

$$x^1 = x = \rho\cos\phi, \qquad u^1 = \rho = \sqrt{x^2+y^2}, \tag{2.1a}$$

$$x^2 = y = \rho\sin\phi, \qquad u^2 = \phi = \tan^{-1}\frac{y}{x}, \tag{2.1b}$$

$$x^3 = z = z, \qquad u^3 = z = z. \tag{2.1c}$$

Let us define the unit vectors

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2.2a}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2.2b}$$

$$\hat{\mathbf{z}} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + \hat{\mathbf{k}}, \tag{2.2c}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are basis vectors in rectangular coordinate system. We will also use the notation $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}^3 = \hat{\mathbf{x}}_3$, to represent these vectors.

(a) Basis vectors:

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i}. \tag{2.3}$$

Show that

$$\mathbf{e}_1 = \hat{\boldsymbol{\rho}}, \qquad \mathbf{e}_2 = \rho\hat{\boldsymbol{\phi}}, \qquad \mathbf{e}_3 = \hat{\mathbf{z}}. \tag{2.4}$$

(b) Reciprocal basis vectors:

$$\mathbf{e}^i = \boldsymbol{\nabla} u^i. \tag{2.5}$$

Show that

$$\mathbf{e}^1 = \hat{\boldsymbol{\rho}}, \qquad \mathbf{e}^2 = \frac{\hat{\boldsymbol{\phi}}}{\rho}, \qquad \mathbf{e}^3 = \hat{\mathbf{z}}. \tag{2.6}$$

(c) Orthonormality: Show that

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j. \tag{2.7}$$

(d) Metric tensor: A line element is defined as

$$d\mathbf{r} = dx^i \hat{\mathbf{x}}_i = du^i \mathbf{e}_i. \tag{2.8}$$

Show that

$$d\mathbf{r}\cdot d\mathbf{r} = du^i du^j g_{ij}, \tag{2.9}$$

where the metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \tag{2.10}$$

Evaluate all the components of g_{ij}.

(e) Completeness relation: Starting from

$$\boldsymbol{\nabla}\,\mathbf{r} = \mathbf{1} \tag{2.11}$$

derive the completeness relation

$$\mathbf{e}^i\,\mathbf{e}_i = \mathbf{1}. \tag{2.12}$$

Express the completeness relation in cylindrical polar coordinates in terms of $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$.

2. (**20 points.**) Transformation of basis vectors.

(a) Let us consider a set of basis vectors $\mathbf{e}_i$, where $i = 1, 2, 3$, and the associated reciprocal basis vectors $\mathbf{e}^i$ that satisfy the completeness relation

$$\mathbf{1} = \mathbf{e}^i\,\mathbf{e}_i = \mathbf{e}^1\mathbf{e}_1 + \mathbf{e}^2\mathbf{e}_2 + \mathbf{e}^3\mathbf{e}_3. \tag{2.13}$$

It is a complete set because an arbitrary vector $\mathbf{A}$ can be expressed in terms of it's projections along the basis vectors in the following way,

$$\mathbf{A} = \mathbf{A}\cdot\mathbf{1} = \mathbf{A}\cdot(\mathbf{e}^i\,\mathbf{e}_i) = (\mathbf{A}\cdot\mathbf{e}^i)\,\mathbf{e}_i = A^i\,\mathbf{e}_i, \tag{2.14}$$

where we recognized and defined the projections of vector $\mathbf{A}$ along the direction of basis vectors as the components

$$A^i = (\mathbf{A}\cdot\mathbf{e}^i). \tag{2.15}$$

Similarly, multiplying by the identity dyadic on the left gives

$$\mathbf{A} = \mathbf{1}\cdot\mathbf{A} = (\mathbf{e}^i\,\mathbf{e}_i)\cdot\mathbf{A} = \mathbf{e}^i\,(\mathbf{e}_i\cdot\mathbf{A}) = \mathbf{e}^i\,A_i, \tag{2.16}$$

where now the projections of vector $\mathbf{A}$ in the direction of the reciprocal basis vectors are the components

$$A_i = (\mathbf{e}_i\cdot\mathbf{A}). \tag{2.17}$$

For consistency we require the equality

$$(\mathbf{A}\cdot\mathbf{e}^i)\,\mathbf{e}_i = \mathbf{e}^i\,(\mathbf{e}_i\cdot\mathbf{A}). \tag{2.18}$$

Thus, derive

$$A^j = g^{ji}A_i, \tag{2.19a}$$
$$A_j = g_{ji}A^i, \tag{2.19b}$$

where the metric tensors are defined as

$$g^{ji} = \mathbf{e}^j\cdot\mathbf{e}^i, \tag{2.20a}$$
$$g_{ji} = \mathbf{e}_j\cdot\mathbf{e}_i. \tag{2.20b}$$

(b) For another set of basis vectors $\mathbf{g}_i$ and the associated reciprocal basis vectors $\mathbf{g}^i$ that satisfy the completeness relation

$$\mathbf{1} = \mathbf{g}^i \, \mathbf{g}_i \tag{2.21}$$

we can write

$$\mathbf{A} = (\mathbf{A} \cdot \mathbf{g}^i) \, \mathbf{g}_i = \bar{A}^i \, \mathbf{g}_i, \tag{2.22}$$

where the components $\bar{A}^i$ are in general different from A^i, and

$$\mathbf{A} = \mathbf{g}^i \, (\mathbf{g}_i \cdot \mathbf{A}) = \mathbf{g}^i \, \bar{A}_i. \tag{2.23}$$

For consistency we require the equality

$$(\mathbf{A} \cdot \mathbf{g}^i) \, \mathbf{g}_i = \mathbf{g}^i \, (\mathbf{g}_i \cdot \mathbf{A}). \tag{2.24}$$

Thus, derive

$$\bar{A}^j = \bar{g}^{ji} \bar{A}_i, \tag{2.25a}$$
$$\bar{A}_j = \bar{g}_{ji} \bar{A}^i, \tag{2.25b}$$

where the metric tensors are defined as

$$\bar{g}^{ji} = \mathbf{g}^j \cdot \mathbf{g}^i, \tag{2.26a}$$
$$\bar{g}_{ji} = \mathbf{g}_j \cdot \mathbf{g}_i. \tag{2.26b}$$

(c) For consistency between the two independent basis vector representations we require the equality

$$(\mathbf{A} \cdot \mathbf{e}^i) \, \mathbf{e}_i = (\mathbf{A} \cdot \mathbf{g}^i) \, \mathbf{g}_i. \tag{2.27}$$

Taking the dot product on the right with $\mathbf{e}^j$ and using orthogonality relation

$$\mathbf{e}_i \cdot \mathbf{e}^i = \delta^i_j \tag{2.28}$$

we obtain

$$(\mathbf{A} \cdot \mathbf{e}^j) = (\mathbf{A} \cdot \mathbf{g}^i) \, [\mathbf{g}_i \cdot \mathbf{e}^j]. \tag{2.29}$$

Similarly, taking the dot product on the right with $\mathbf{g}^j$ we obtain

$$(\mathbf{A} \cdot \mathbf{e}^i) \, [\mathbf{e}_i \cdot \mathbf{g}^j] = (\mathbf{A} \cdot \mathbf{g}^j). \tag{2.30}$$

In terms of the transformation matrices connecting the two basis vectors,

$$S_i{}^j = [\mathbf{g}_i \cdot \mathbf{e}^j] \tag{2.31}$$

and

$$R_i{}^j = [\mathbf{e}_i \cdot \mathbf{g}^j], \tag{2.32}$$

we can derive the transformation of the components

$$A^j = \bar{A}^i S_i{}^j, \tag{2.33a}$$
$$A^i R_i{}^j = \bar{A}^j. \tag{2.33b}$$

Further, derive

$$A_j = R_j{}^i \bar{A}_i, \tag{2.34a}$$
$$S_j{}^i A_i = \bar{A}^j. \tag{2.34b}$$

Show that

$$R_i{}^j S_j{}^k = [\mathbf{e}_i \cdot \mathbf{g}^j][\mathbf{g}_j \cdot \mathbf{e}^k] = \mathbf{e}_i \cdot (\mathbf{g}^j \, \mathbf{g}_j) \cdot \mathbf{e}^k = \mathbf{e}_i \cdot \mathbf{1} \cdot \mathbf{e}^k = \delta^k_i. \tag{2.35}$$

(d) i. Find the transformation matrices S and T between cylindrical polar coordinates and rectangular coordinates. Verify that $ST = 1$.

ii. Find the transformation matrices S and T between cylindrical polar coordinates and spherical polar coordinates. Verify that $ST = 1$.

3. (**Example.**) Let $\mathbf{r}$ represent the position vector, x^i the components of the position vector in rectangular coordinates, and u^i the components of the position vector in spherical polar coordinates. In particular, we have

$$x^1 = x = r\sin\theta\cos\phi, \qquad u^1 = r = \sqrt{x^2+y^2+z^2}, \tag{2.36a}$$

$$x^2 = y = r\sin\theta\sin\phi, \qquad u^2 = \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \tag{2.36b}$$

$$x^3 = z = r\cos\theta, \qquad u^3 = \phi = \tan^{-1}\frac{y}{x}. \tag{2.36c}$$

Let us define the unit vectors

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}, \tag{2.37a}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}, \tag{2.37b}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}, \tag{2.37c}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are basis vectors in rectangular coordinate system. We will also use the notation $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}^3 = \hat{\mathbf{x}}_3$, to represent these vectors.

(a) Basis vectors:

$$\mathbf{e}_i = \frac{\partial\mathbf{r}}{\partial u^i}. \tag{2.38}$$

Show that

$$\mathbf{e}_1 = \hat{\mathbf{r}}, \qquad \mathbf{e}_2 = r\hat{\boldsymbol{\theta}}, \qquad \mathbf{e}_3 = r\sin\theta\hat{\boldsymbol{\phi}}. \tag{2.39}$$

(b) Reciprocal basis vectors:

$$\mathbf{e}^i = \boldsymbol{\nabla}u^i. \tag{2.40}$$

Show that

$$\mathbf{e}^1 = \hat{\mathbf{r}}, \qquad \mathbf{e}^2 = \frac{\hat{\boldsymbol{\theta}}}{r}, \qquad \mathbf{e}^3 = \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}. \tag{2.41}$$

(c) Orthonormality: Show that

$$\mathbf{e}^i\cdot\mathbf{e}_j = \delta^i_j. \tag{2.42}$$

(d) Metric tensor: A line element is defined as

$$d\mathbf{r} = dx^i\hat{\mathbf{x}}_i = du^i\mathbf{e}_i. \tag{2.43}$$

Show that

$$d\mathbf{r}\cdot d\mathbf{r} = du^i du^j g_{ij}, \tag{2.44}$$

where the metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i\cdot\mathbf{e}_j. \tag{2.45}$$

Evaluate all the components of g_{ij}.

(e) Completeness relation: Starting from

$$\boldsymbol{\nabla}\mathbf{r} = \mathbf{1} \tag{2.46}$$

derive the completeness relation

$$\mathbf{e}^i\mathbf{e}_i = \mathbf{1}. \tag{2.47}$$

Express the completeness relation in spherical polar coordinates in terms of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$.

(f) Transformation matrix: The components of a vector $\mathbf{A}$ are defined using the relations

$$\mathbf{A} = \mathbf{A}\cdot\mathbf{1} = A^i\hat{\mathbf{x}}_i = \bar{A}^i\mathbf{e}_i, \tag{2.48a}$$
$$\mathbf{A} = \mathbf{1}\cdot\mathbf{A} = \hat{\mathbf{x}}^i A_i = \mathbf{e}^i\bar{A}_i. \tag{2.48b}$$

Then, derive the transformations

$$\bar{A}^j = A^i T_i{}^j, \qquad T_i{}^j = \hat{\mathbf{x}}_i\cdot\mathbf{e}^j, \tag{2.49a}$$
$$\bar{A}_j = S_j{}^i A_i, \qquad S_j{}^i = \mathbf{e}_j\cdot\hat{\mathbf{x}}^i, \tag{2.49b}$$

and show that $T_i{}^j S_j{}^k = \delta_i^k$. Find S and T for spherical polar coordinates.

4. (**Example.**) Let $\mathbf{r}$ represent a position vector in three dimensional space. Let x^i be the components of the position vector in rectangular coordinates, which can be interpreted as surfaces of constant x^i. Let us coordinatize the space using the planes, labeled using β,

$$y = mx + \beta \tag{2.50}$$

where m is fixed, instead of planes with constant y. The other two sets of planes of constant x and constant z are the same. See Fig. 2.1. Let u^i be the components of the position vector in this new coordinatization of space. In particular, we have

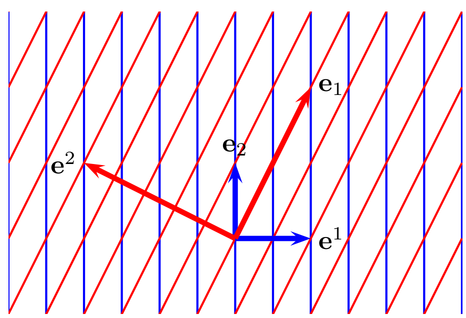

Figure 2.1: Basis vectors $\mathbf{e}_i$ and reciprocal basis vectors $\mathbf{e}^i$.

$$x^1 = x = \alpha, \qquad u^1 = \alpha = x, \tag{2.51a}$$
$$x^2 = y = mx + \beta, \qquad u^2 = \beta = y - mx, \tag{2.51b}$$
$$x^3 = z = \gamma, \qquad u^3 = \gamma = z. \tag{2.51c}$$

The basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ in rectangular coordinate system will be represented as $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}^3 = \hat{\mathbf{x}}_3$, if necessary.

(a) Basis vectors:

$$\mathbf{e}_i = \frac{\partial\mathbf{r}}{\partial u^i}. \tag{2.52}$$

Show that

$$\mathbf{e}_1 = \hat{\mathbf{i}} + m\,\hat{\mathbf{j}}, \qquad \mathbf{e}_2 = \hat{\mathbf{j}}, \qquad \mathbf{e}_3 = \hat{\mathbf{k}}. \tag{2.53}$$

(b) Reciprocal basis vectors:

$$\mathbf{e}^i = \boldsymbol{\nabla} u^i. \tag{2.54}$$

Show that

$$\mathbf{e}^1 = \hat{\mathbf{i}}, \qquad \mathbf{e}^2 = -m\,\hat{\mathbf{i}} + \hat{\mathbf{j}}, \qquad \mathbf{e}^3 = \hat{\mathbf{k}}. \tag{2.55}$$

Verify the relations

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{(\mathbf{e}_2 \times \mathbf{e}_3) \cdot \mathbf{e}_1}, \qquad \mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{(\mathbf{e}_3 \times \mathbf{e}_1) \cdot \mathbf{e}_2}, \qquad \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3}. \tag{2.56}$$

(c) Orthonormality: Show that

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j. \tag{2.57}$$

That is,

$$\mathbf{e}^1 \cdot \mathbf{e}_1 = 1, \qquad \mathbf{e}^1 \cdot \mathbf{e}_2 = 0, \qquad \mathbf{e}^1 \cdot \mathbf{e}_3 = 0, \tag{2.58a}$$
$$\mathbf{e}^2 \cdot \mathbf{e}_1 = 0, \qquad \mathbf{e}^2 \cdot \mathbf{e}_2 = 1, \qquad \mathbf{e}^2 \cdot \mathbf{e}_3 = 0, \tag{2.58b}$$
$$\mathbf{e}^3 \cdot \mathbf{e}_1 = 0, \qquad \mathbf{e}^3 \cdot \mathbf{e}_2 = 0, \qquad \mathbf{e}^3 \cdot \mathbf{e}_3 = 1. \tag{2.58c}$$

(d) Metric tensor: The metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \tag{2.59}$$

Evaluate all the components of g_{ij}. That is,

$$g_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1 + m^2, \qquad g_{12} = \mathbf{e}_1 \cdot \mathbf{e}_2 = m, \qquad g_{13} = \mathbf{e}_1 \cdot \mathbf{e}_3 = 0, \tag{2.60a}$$
$$g_{21} = \mathbf{e}_2 \cdot \mathbf{e}_1 = m, \qquad g_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1, \qquad g_{23} = \mathbf{e}_2 \cdot \mathbf{e}_3 = 0, \tag{2.60b}$$
$$g_{31} = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0, \qquad g_{32} = \mathbf{e}_3 \cdot \mathbf{e}_2 = 0, \qquad g_{33} = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1. \tag{2.60c}$$

Similarly evaluate the components of

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j. \tag{2.61}$$

That is,

$$g^{11} = \mathbf{e}^1 \cdot \mathbf{e}^1 = 1, \qquad g^{12} = \mathbf{e}^1 \cdot \mathbf{e}^2 = -m, \qquad g^{13} = \mathbf{e}^1 \cdot \mathbf{e}^3 = 0, \tag{2.62a}$$
$$g^{21} = \mathbf{e}^2 \cdot \mathbf{e}^1 = -m, \qquad g^{22} = \mathbf{e}^2 \cdot \mathbf{e}^2 = 1 + m^2, \qquad g^{23} = \mathbf{e}^2 \cdot \mathbf{e}^3 = 0, \tag{2.62b}$$
$$g^{31} = \mathbf{e}^3 \cdot \mathbf{e}^1 = 0, \qquad g^{32} = \mathbf{e}^3 \cdot \mathbf{e}^2 = 0, \qquad g^{33} = \mathbf{e}^3 \cdot \mathbf{e}^3 = 1. \tag{2.62c}$$

Verify that $g^{ij} g_{jk} = \delta^i_k$.

(e) Completeness relation: Verify the completeness relation

$$\mathbf{e}^i \mathbf{e}_i = \mathbf{1} \tag{2.63}$$

by evaluating

$$\mathbf{e}^1 \mathbf{e}_1 + \mathbf{e}^2 \mathbf{e}_2 + \mathbf{e}^3 \mathbf{e}_3. \tag{2.64}$$

(f) Given a vector

$$\mathbf{A} = a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}} \tag{2.65}$$

in rectangular coordinates, find the components of the vector $\mathbf{A}$ in the basis of $\mathbf{e}_i$. That is, find the components A^i in

$$\mathbf{A} = A^1\,\mathbf{e}_1 + A^2\,\mathbf{e}_2 + A^3\,\mathbf{e}_3. \tag{2.66}$$

5. (**20 points.**) In terms of the unit vectors

$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2.67a}$$
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2.67b}$$
$$\hat{\mathbf{z}} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + \hat{\mathbf{k}}, \tag{2.67c}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are basis vectors in rectangular coordinate system the basis vectors in cylindrical polar coordinates are

$$\mathbf{e}_1 = \hat{\boldsymbol{\rho}}, \qquad \mathbf{e}_2 = \rho\hat{\boldsymbol{\phi}}, \qquad \mathbf{e}_3 = \hat{\mathbf{z}}. \tag{2.68}$$

Evaluate all the components of the metric tensor

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \tag{2.69}$$

6. (**Half-Sinusoidal coordinates.**) Let $\mathbf{r}$ represent a position vector in three dimensional space. Let x^i be the components of the position vector in rectangular coordinates, which can be interpreted as surfaces of constant x^i. Let us coordinatize the space using a sinusoidally corrugated planes labeled using t, and two planes, labeled using s and x,

$$t = z - h\sin ky, \tag{2.70a}$$
$$s = y, \tag{2.70b}$$
$$x = x, \tag{2.70c}$$

where h is the amplitude of the corrugations and $\lambda = 2\pi/k$ is the wavelength of the corrugations. See Fig. 2.2. Let u^i be the components of the position vector in this new coordinatization of space. In particular, we have

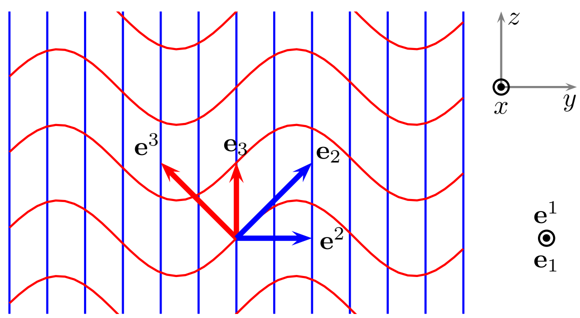

Figure 2.2: Basis vectors $\mathbf{e}_i$ and reciprocal basis vectors $\mathbf{e}^i$.

$$x^1 = x = x, \qquad u^1 = x = x, \tag{2.71a}$$
$$x^2 = y = s, \qquad u^2 = s = y, \tag{2.71b}$$
$$x^3 = z = t + h\sin ks, \qquad u^3 = t = z - h\sin ky. \tag{2.71c}$$

The basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ in rectangular coordinate system will be represented as $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}^3 = \hat{\mathbf{x}}_3$, if necessary. We shall use the short hand notation

$$\alpha = hk\cos ky = hk\cos ks. \tag{2.72}$$

(a) Tangent vectors:

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i}. \tag{2.73}$$

Show that

$$\mathbf{e}_1 = \hat{\mathbf{i}}, \qquad \mathbf{e}_2 = \hat{\mathbf{j}} + \alpha\hat{\mathbf{k}}, \qquad \mathbf{e}_3 = \hat{\mathbf{k}}. \tag{2.74}$$

(b) Normal vectors:

$$\mathbf{e}^i = \boldsymbol{\nabla} u^i. \tag{2.75}$$

Show that

$$\mathbf{e}^1 = \hat{\mathbf{i}}, \qquad \mathbf{e}^2 = \hat{\mathbf{j}}, \qquad \mathbf{e}^3 = -\alpha\,\hat{\mathbf{j}} + \hat{\mathbf{k}}. \tag{2.76}$$

Verify the relations

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk}\mathbf{e}^k, \tag{2.77}$$
$$\mathbf{e}^i \times \mathbf{e}^j = \varepsilon^{ijk}\mathbf{e}_k. \tag{2.78}$$

(c) Orthonormality: Show that

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j. \tag{2.79}$$

That is,

$$\mathbf{e}^1 \cdot \mathbf{e}_1 = 1, \qquad \mathbf{e}^1 \cdot \mathbf{e}_2 = 0, \qquad \mathbf{e}^1 \cdot \mathbf{e}_3 = 0, \tag{2.80a}$$
$$\mathbf{e}^2 \cdot \mathbf{e}_1 = 0, \qquad \mathbf{e}^2 \cdot \mathbf{e}_2 = 1, \qquad \mathbf{e}^2 \cdot \mathbf{e}_3 = 0, \tag{2.80b}$$
$$\mathbf{e}^3 \cdot \mathbf{e}_1 = 0, \qquad \mathbf{e}^3 \cdot \mathbf{e}_2 = 0, \qquad \mathbf{e}^3 \cdot \mathbf{e}_3 = 1. \tag{2.80c}$$

(d) Metric tensor: The metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \tag{2.81}$$

Evaluate all the components of g_{ij}. That is,

$$g_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1, \qquad g_{12} = \mathbf{e}_1 \cdot \mathbf{e}_2 = 0, \qquad g_{13} = \mathbf{e}_1 \cdot \mathbf{e}_3 = 0, \tag{2.82a}$$
$$g_{21} = \mathbf{e}_2 \cdot \mathbf{e}_1 = 0, \qquad g_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1 + \alpha^2, \qquad g_{23} = \mathbf{e}_2 \cdot \mathbf{e}_3 = \alpha, \tag{2.82b}$$
$$g_{31} = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0, \qquad g_{32} = \mathbf{e}_3 \cdot \mathbf{e}_2 = \alpha, \qquad g_{33} = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1. \tag{2.82c}$$

Similarly evaluate the components of

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j. \tag{2.83}$$

That is,

$$g^{11} = \mathbf{e}^1 \cdot \mathbf{e}^1 = 1, \qquad g^{12} = \mathbf{e}^1 \cdot \mathbf{e}^2 = 0, \qquad g^{13} = \mathbf{e}^1 \cdot \mathbf{e}^3 = 0, \tag{2.84a}$$
$$g^{21} = \mathbf{e}^2 \cdot \mathbf{e}^1 = 0, \qquad g^{22} = \mathbf{e}^2 \cdot \mathbf{e}^2 = 1, \qquad g^{23} = \mathbf{e}^2 \cdot \mathbf{e}^3 = -\alpha, \tag{2.84b}$$
$$g^{31} = \mathbf{e}^3 \cdot \mathbf{e}^1 = 0, \qquad g^{32} = \mathbf{e}^3 \cdot \mathbf{e}^2 = -\alpha, \qquad g^{33} = \mathbf{e}^3 \cdot \mathbf{e}^3 = 1 + \alpha^2. \tag{2.84c}$$

Verify that $g^{ij}g_{jk} = \delta^i_k$.

(e) Completeness relation: Verify the completeness relation

$$\mathbf{e}^i\mathbf{e}_i = \mathbf{1} \tag{2.85}$$

by evaluating

$$\mathbf{e}^1\mathbf{e}_1 + \mathbf{e}^2\mathbf{e}_2 + \mathbf{e}^3\mathbf{e}_3. \tag{2.86}$$

(f) Given a vector

$$\mathbf{A} = a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}} + c\,\hat{\mathbf{k}} \tag{2.87}$$

in rectangular coordinates, find the components of the vector $\mathbf{A}$ in the basis of $\mathbf{e}_i$. That is, find the components A^i in

$$\mathbf{A} = A^1\,\mathbf{e}_1 + A^2\,\mathbf{e}_2 + A^3\,\mathbf{e}_3. \tag{2.88}$$

7. (**Sinusoidal coordinates.**) Let $\mathbf{r}$ represent a position vector in two dimensional space. Let x^i be the components of the position vector in rectangular coordinates, which can be interpreted as surfaces of constant x^i. Let us coordinatize the space using sinusoidally corrugated planes labeled using x' and y',

$$x = x' - h\sin ky, \tag{2.89a}$$
$$y = y' + h\sin kx, \tag{2.89b}$$
$$z = z', \tag{2.89c}$$

where h is the amplitude of the corrugations and $\lambda = 2\pi/k$ is the wavelength of the corrugations. See Fig. 2.3. Let u^i be the components of the position vector in this new coordinatization of space. In particular, we have

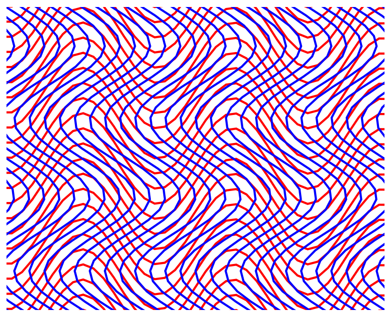

Figure 2.3: Coordinate chart using sinusoidal coordinates.

$$x^1 = x = x' - h\sin ky, \qquad u^1 = x' = x + h\sin ky, \tag{2.90a}$$
$$x^2 = y = y' + h\sin kx, \qquad u^2 = y' = y - h\sin kx. \tag{2.90b}$$

The basis vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in rectangular coordinate system will be represented as $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, if necessary.

(a) Tangent vectors:

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i}. \tag{2.91}$$

Show that

$$\mathbf{e}_1 = \frac{\hat{\mathbf{i}} + \hat{\mathbf{j}}\, kh\cos kx}{1 + k^2h^2\cos kx\cos ky}, \tag{2.92a}$$
$$\mathbf{e}_2 = \frac{-\hat{\mathbf{i}}\, kh\cos ky + \hat{\mathbf{j}}}{1 + k^2h^2\cos kx\cos ky}. \tag{2.92b}$$

(b) Normal vectors:

$$\mathbf{e}^i = \boldsymbol{\nabla} u^i. \tag{2.93}$$

Show that

$$\mathbf{e}^1 = \hat{\mathbf{i}} + \hat{\mathbf{j}}\, kh\cos ky, \tag{2.94a}$$
$$\mathbf{e}^2 = -\hat{\mathbf{i}}\, kh\cos kx + \hat{\mathbf{j}}. \tag{2.94b}$$

(c) Orthonormality: Show that

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j. \tag{2.95}$$

That is,

$$\mathbf{e}^1 \cdot \mathbf{e}_1 = 1, \qquad \mathbf{e}^1 \cdot \mathbf{e}_2 = 0, \tag{2.96a}$$
$$\mathbf{e}^2 \cdot \mathbf{e}_1 = 0, \qquad \mathbf{e}^2 \cdot \mathbf{e}_2 = 1. \tag{2.96b}$$

(d) Metric tensor: The metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \tag{2.97}$$

Evaluate all the components of g_{ij}. That is,

$$g_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1 = \frac{1 + k^2h^2\cos^2 kx}{(1 + k^2h^2\cos kx\cos ky)^2}, \qquad g_{12} = \mathbf{e}_1 \cdot \mathbf{e}_2 = \frac{kh\cos kx - kh\cos ky}{(1 + k^2h^2\cos kx\cos ky)^2}, \tag{2.98a}$$
$$g_{21} = \mathbf{e}_2 \cdot \mathbf{e}_1 = \frac{kh\cos kx - kh\cos ky}{(1 + k^2h^2\cos kx\cos ky)^2}, \qquad g_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2 = \frac{1 + k^2h^2\cos^2 ky}{(1 + k^2h^2\cos kx\cos ky)^2}. \tag{2.98b}$$

Similarly evaluate the components of

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j. \tag{2.99}$$

That is,

$$g^{11} = \mathbf{e}^1 \cdot \mathbf{e}^1 = 1 + k^2h^2\cos^2 ky, \qquad g^{12} = \mathbf{e}^1 \cdot \mathbf{e}^2 = -(kh\cos kx - kh\cos ky), \tag{2.100a}$$
$$g^{21} = \mathbf{e}^2 \cdot \mathbf{e}^1 = -(kh\cos kx - kh\cos ky), \qquad g^{22} = \mathbf{e}^2 \cdot \mathbf{e}^2 = 1 + k^2h^2\cos^2 kx. \tag{2.100b}$$

Verify that $g^{ij}g_{jk} = \delta^i_k$.

(e) Completeness relation: Verify the completeness relation

$$\mathbf{e}^i\mathbf{e}_i = \mathbf{1} \tag{2.101}$$

by evaluating

$$\mathbf{e}^1\mathbf{e}_1 + \mathbf{e}^2\mathbf{e}_2. \tag{2.102}$$

8. (**20 points.**) Is the coordinate chart using sinusoidal coordinates in Fig. 2.4 well defined?

   (a) Find the points where the tangent vectors align–what is the implication.

   (b) The intersection of a curve of fixed x' and a curve of fixed y' is multivalued. Thus, is this a well-defined chart?

   (c) Is there a doubly periodic minimal curve?

2.2 Connection, Christoffel symbols

The connection is defined as

$$(\boldsymbol{\nabla}\mathbf{e}_i). \tag{2.103}$$

Berry connection $\mathbf{A}_i{}^k$ captures the projections of the connection to the right,

$$\mathbf{A}_i{}^k = (\boldsymbol{\nabla}\mathbf{e}_i) \cdot \mathbf{e}^k = \left(\boldsymbol{\nabla}\frac{\partial \mathbf{r}}{\partial u^i}\right) \cdot \frac{\partial u^k}{\partial \mathbf{r}}. \tag{2.104}$$

Similarly we can define

$$\mathbf{A}^k{}_i = (\boldsymbol{\nabla}\mathbf{e}^k) \cdot \mathbf{e}_i = \left(\boldsymbol{\nabla}\frac{\partial u^k}{\partial \mathbf{r}}\right) \cdot \frac{\partial \mathbf{r}}{\partial u^i}. \tag{2.105}$$

Taking the gradient of the identity $\mathbf{e}_i \cdot \mathbf{e}^k = \delta^k_i$, we observe

$$(\boldsymbol{\nabla}\mathbf{e}_i) \cdot \mathbf{e}^k + (\boldsymbol{\nabla}\mathbf{e}^k) \cdot \mathbf{e}_i = 0, \tag{2.106}$$

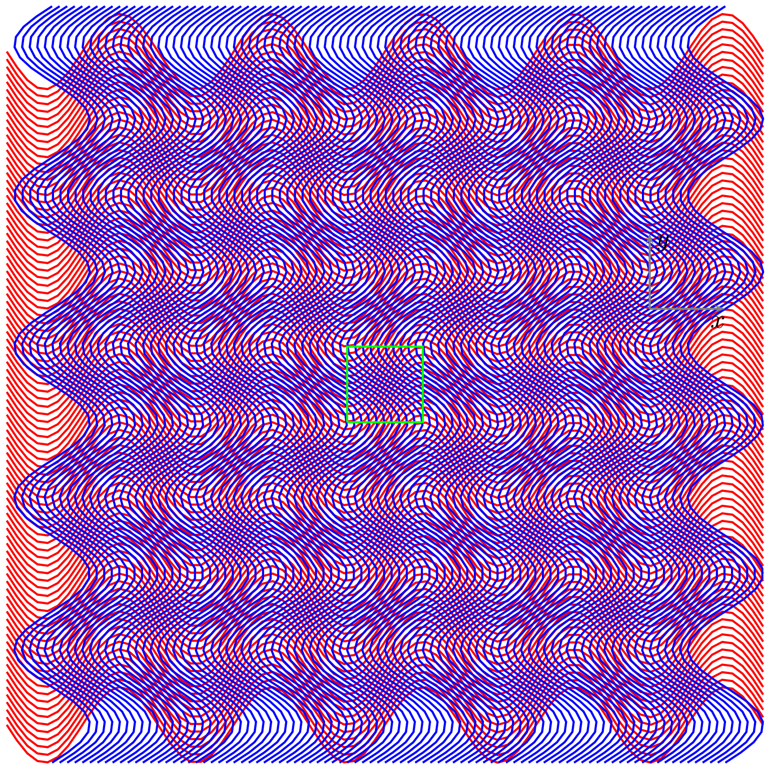

Figure 2.4: Coordinate chart using sinusoidal coordinates.

and deduce the antisymmetric property

$$\mathbf{A}_i{}^k = -\mathbf{A}^k{}_i. \tag{2.107}$$

The projections of the connection to the left

$$\mathbf{A}_{ij} = \mathbf{e}_j \cdot \boldsymbol{\nabla} \mathbf{e}_i = \frac{\partial \mathbf{e}_i}{\partial u^j} \tag{2.108}$$

captures the changes in the basis vectors $\mathbf{e}_i$ in the direction of $\mathbf{e}_j$. The Christoffel symbols Γ_{ij}^k captures all the projections of the connection,

$$\mathbf{e}_j \cdot (\boldsymbol{\nabla} \mathbf{e}_i) \cdot \mathbf{e}^k = \left(\frac{\partial}{\partial u^j} \mathbf{e}_i \right) \cdot \mathbf{e}^k = \mathbf{A}_{ij} \cdot \mathbf{e}^k = \Gamma_{ij}^k. \tag{2.109}$$

1. (**10 points.**) The contravariant basis vectors are

$$\mathbf{e}^i = \boldsymbol{\nabla} u^i. \tag{2.110}$$

Using the completeness relation we have

$$\mathbf{e}^i = \mathbf{1} \cdot \boldsymbol{\nabla} u^i = \mathbf{e}^k \mathbf{e}_k \cdot \boldsymbol{\nabla} u^i. \tag{2.111}$$

Consistency, then, requires that

$$\mathbf{e}_k \cdot \boldsymbol{\nabla} u^i = \delta_k{}^i. \tag{2.112}$$

This suggests the identification

$$\mathbf{e}_k \cdot \boldsymbol{\nabla} = \frac{\partial}{\partial u^k}. \tag{2.113}$$

2. (**20 points.**) [Cylindrical coordinates] The tangent and normal vectors for the cylindrical coordinate system are

$$\mathbf{e}_1 = \mathbf{e}_\rho = \hat{\boldsymbol{\rho}}, \qquad \mathbf{e}^1 = \mathbf{e}^\rho = \hat{\boldsymbol{\rho}}, \tag{2.114a}$$

$$\mathbf{e}_2 = \mathbf{e}_\phi = \rho\hat{\boldsymbol{\phi}}, \qquad \mathbf{e}^2 = \mathbf{e}^\phi = \frac{\hat{\boldsymbol{\phi}}}{\rho}, \tag{2.114b}$$

$$\mathbf{e}_3 = \mathbf{e}_z = \hat{\mathbf{z}}, \qquad \mathbf{e}^3 = \mathbf{e}^z = \hat{\mathbf{z}}. \tag{2.114c}$$

Compute the Berry connection for the cylindrical coordinate system to be

$$\mathbf{A}_{ij} = \begin{pmatrix} 0 & \hat{\boldsymbol{\phi}} & 0 \\ \hat{\boldsymbol{\phi}} & -\rho\hat{\boldsymbol{\rho}} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\mathbf{e}_2}{\rho} & 0 \\ \frac{\mathbf{e}_2}{\rho} & -\rho\mathbf{e}_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.115}$$

Compute the Christoffel symbols for the cylindrical coordinate system. Show that the non-zero Christoffel symbols in cylindrical coordinates are

$$\Gamma^1_{22} = \Gamma^\rho_{\phi\phi} = -\rho, \tag{2.116}$$

$$\Gamma^2_{12} = \Gamma^\phi_{\rho\phi} = \frac{1}{\rho}, \tag{2.117}$$

$$\Gamma^2_{21} = \Gamma^\phi_{\phi\rho} = \frac{1}{\rho}. \tag{2.118}$$

3. (**20 points.**) [Cylindrical coordinates] The tangent and normal vectors for the cylindrical coordinate system are

$$\mathbf{e}_1 = \mathbf{e}_\rho = \hat{\boldsymbol{\rho}}, \qquad \mathbf{e}^1 = \mathbf{e}^\rho = \hat{\boldsymbol{\rho}}, \tag{2.119a}$$

$$\mathbf{e}_2 = \mathbf{e}_\phi = \rho\hat{\boldsymbol{\phi}}, \qquad \mathbf{e}^2 = \mathbf{e}^\phi = \frac{\hat{\boldsymbol{\phi}}}{\rho}, \tag{2.119b}$$

$$\mathbf{e}_3 = \mathbf{e}_z = \hat{\mathbf{z}}, \qquad \mathbf{e}^3 = \mathbf{e}^z = \hat{\mathbf{z}}. \tag{2.119c}$$

The connection is defined as

$$(\boldsymbol{\nabla}\mathbf{e}_i). \tag{2.120}$$

Berry connection ${\mathbf{A}_i}^k$ captures the projections of the connection to the right,

$${\mathbf{A}_i}^k = (\boldsymbol{\nabla}\mathbf{e}_i)\cdot\mathbf{e}^k. \tag{2.121}$$

Compute the Berry connection ${\mathbf{A}_i}^k$ for the cylindrical coordinate system to be

$${\mathbf{A}_i}^k = \begin{pmatrix} 0 & \frac{\hat{\boldsymbol{\phi}}}{\rho^2} & 0 \\ -\hat{\boldsymbol{\phi}} & \frac{\boldsymbol{\rho}}{\rho^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.122}$$

The Christoffel symbols Γ^k_{ij} captures all the projections of the connection,

$$\mathbf{e}_j\cdot(\boldsymbol{\nabla}\mathbf{e}_i)\cdot\mathbf{e}^k = \Gamma^k_{ij}. \tag{2.123}$$

Compute the Christoffel symbols for the cylindrical coordinate system. Show that the non-zero Christoffel symbols in cylindrical coordinates are

$$\Gamma^1_{22} = \Gamma^\rho_{\phi\phi} = -\rho, \tag{2.124}$$

$$\Gamma^2_{12} = \Gamma^\phi_{\rho\phi} = \frac{1}{\rho}, \tag{2.125}$$

$$\Gamma^2_{21} = \Gamma^\phi_{\phi\rho} = \frac{1}{\rho}. \tag{2.126}$$

4. (**10 points.**) [Spherical coordinates] In term of unit vectors

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}, \tag{2.127a}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}, \tag{2.127b}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}, \tag{2.127c}$$

the tangent and normal vectors for spherical polar coordinates are

$$\mathbf{e}_1 = \mathbf{e}_r = \hat{\mathbf{r}} \qquad \mathbf{e}^1 = \mathbf{e}^r = \hat{\mathbf{r}}, \tag{2.128a}$$

$$\mathbf{e}_2 = \mathbf{e}_\theta = r\hat{\boldsymbol{\theta}} \qquad \mathbf{e}^2 = \mathbf{e}^\theta = \frac{1}{r}\hat{\boldsymbol{\theta}}, \tag{2.128b}$$

$$\mathbf{e}_3 = \mathbf{e}_\phi = r\sin\theta\hat{\boldsymbol{\phi}} \qquad \mathbf{e}^3 = \mathbf{e}^\phi = \frac{1}{r\sin\theta}\hat{\boldsymbol{\phi}}. \tag{2.128c}$$

Compute the following projections of the connection for the spherical coordinate system to be

$$\mathbf{A}_{ij} = \begin{pmatrix} 0 & \hat{\boldsymbol{\theta}} & \sin\theta\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{\theta}} & -r\hat{\mathbf{r}} & r\cos\theta\hat{\boldsymbol{\phi}} \\ \sin\theta\hat{\boldsymbol{\phi}} & r\cos\theta\hat{\boldsymbol{\phi}} & -r\sin\theta(\sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}} \end{pmatrix} \tag{2.129a}$$

$$= \begin{pmatrix} 0 & \frac{\hat{\mathbf{e}}_\theta}{r} & \frac{\hat{\mathbf{e}}_\phi}{r} \\ \frac{\hat{\mathbf{e}}_\theta}{r} & -r\hat{\mathbf{e}}_r & \cot\theta\hat{\mathbf{e}}_\phi \\ \frac{\hat{\mathbf{e}}_\phi}{r} & \cot\theta\hat{\mathbf{e}}_\phi & -r\sin^2\theta\hat{\mathbf{e}}_r - \sin\theta\cos\theta\hat{\mathbf{e}}_\theta \end{pmatrix}. \tag{2.129b}$$

Compute the Christoffel symbols

$$\Gamma^k_{ij} = \mathbf{A}_{ij}\cdot\mathbf{e}^k \tag{2.130}$$

for the spherical coordinate system.

5. (**10 points.**) [Sinusoidal coordinates] The tangent and normal vectors for sinusoidal coordinates, introduced in Section 2.1 Item 6, are

$$\mathbf{e}_1 = \hat{\mathbf{i}} \qquad \mathbf{e}^1 = \hat{\mathbf{i}}, \tag{2.131a}$$

$$\mathbf{e}_2 = \hat{\mathbf{j}} + \alpha\hat{\mathbf{k}} \qquad \mathbf{e}^2 = \hat{\mathbf{j}}, \tag{2.131b}$$

$$\mathbf{e}_3 = \hat{\mathbf{k}} \qquad \mathbf{e}^3 = -\alpha\hat{\mathbf{j}} + \mathbf{k}, \tag{2.131c}$$

where $\alpha = hk\cos ks = hk\cos hy$. Compute the following projections of the connection for the sinusoidal coordinate system to be

$$\mathbf{A}_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -k\alpha\hat{\mathbf{e}}_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.132}$$

Compute the Christoffel symbols

$$\Gamma^k_{ij} = \mathbf{A}_{ij}\cdot\mathbf{e}^k \tag{2.133}$$

for the spherical coordinate system.

2.3 Curvature

Curvature is a measure of the change in change in the basis vectors,

$$\boldsymbol{\nabla}\boldsymbol{\nabla}\mathbf{e}_i. \tag{2.134}$$

Berry curvature is defined as

$$\boldsymbol{\Omega}_i{}^k = \boldsymbol{\nabla} \times \mathbf{A}_i{}^k. \tag{2.135}$$

Riemann curvature is defined as

$$\boldsymbol{\Omega}_{bci}{}^k = (\boldsymbol{\Omega}_i{}^k)_a \varepsilon_{abc} = \boldsymbol{\nabla}_b (\mathbf{A}_i{}^k)_c - \boldsymbol{\nabla}_c (\mathbf{A}_i{}^k)_b. \tag{2.136}$$

1. (**10 points.**)

2.4 Vector calculus in cylindrical polar coordinates

1. (**10 points.**) In cylindrical polar coordinates a point in space is coordinatized by the intersection of family of right circular cylinders, half-planes, and planes, given by

$$\rho = \sqrt{x^2 + y^2}, \tag{2.137a}$$
$$\phi = \tan^{-1}\frac{y}{x}, \tag{2.137b}$$
$$z = z, \tag{2.137c}$$

respectively. Show that the gradient of these surfaces are given by

$$\boldsymbol{\nabla}\rho = \hat{\boldsymbol{\rho}}, \qquad \hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2.138a}$$
$$\boldsymbol{\nabla}\phi = \hat{\boldsymbol{\phi}}, \qquad \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{2.138b}$$
$$\boldsymbol{\nabla}z = \hat{\mathbf{z}}, \qquad \hat{\mathbf{z}} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + \hat{\mathbf{k}}, \tag{2.138c}$$

which are normal to the respective surfaces. Sketch the surfaces and the corresponding normal vectors. This illustrates that $\boldsymbol{\nabla}$(surface) is a vector (field) normal to the surface.

2. (**10 points.**) The action of the gradient operator in cylindrical polar coordinates,

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \tag{2.139}$$

will involve the derivatives of the unit vectors in cylindrical polar coordinates. Evaluate the following

$$\frac{\partial}{\partial\rho}\hat{\boldsymbol{\rho}} = 0, \qquad \frac{\partial}{\partial\rho}\hat{\boldsymbol{\phi}} = 0, \qquad \frac{\partial}{\partial\rho}\hat{\mathbf{z}} = 0, \tag{2.140a}$$
$$\frac{\partial}{\partial\phi}\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}, \qquad \frac{\partial}{\partial\phi}\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\rho}}, \qquad \frac{\partial}{\partial\phi}\hat{\mathbf{z}} = 0, \tag{2.140b}$$
$$\frac{\partial}{\partial z}\hat{\boldsymbol{\rho}} = 0, \qquad \frac{\partial}{\partial z}\hat{\boldsymbol{\phi}} = 0, \qquad \frac{\partial}{\partial z}\hat{\mathbf{z}} = 0. \tag{2.140c}$$

Visualize the above variational statements graphically.

3. (**10 points.**) Evaluate the following divergence of vector fields.

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{\rho}}, \quad \boldsymbol{\nabla}\cdot\hat{\boldsymbol{\phi}}, \quad \boldsymbol{\nabla}\cdot\hat{\mathbf{z}}, \tag{2.141a}$$
$$\boldsymbol{\nabla}\cdot(\rho^2\hat{\boldsymbol{\rho}}), \quad \boldsymbol{\nabla}\cdot(\rho^2\hat{\boldsymbol{\phi}}), \quad \boldsymbol{\nabla}\cdot(\rho^2\hat{\mathbf{z}}), \tag{2.141b}$$
$$\boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right), \quad \boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right), \quad \boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{z}}}{\rho}\right). \tag{2.141c}$$

Draw the vector fields. Visualize and interpret the action of the divergence operator. Which of the above are divergenceless.

4. (**10 points.**) Evaluate the following curl of vector fields.

$$\boldsymbol{\nabla}\times\hat{\boldsymbol{\rho}},\quad \boldsymbol{\nabla}\times\hat{\boldsymbol{\phi}},\quad \boldsymbol{\nabla}\times\hat{\mathbf{z}}, \tag{2.142a}$$

$$\boldsymbol{\nabla}\times(\rho^2\hat{\boldsymbol{\rho}}),\quad \boldsymbol{\nabla}\times(\rho^2\hat{\boldsymbol{\phi}}),\quad \boldsymbol{\nabla}\times(\rho^2\hat{\mathbf{z}}), \tag{2.142b}$$

$$\boldsymbol{\nabla}\times\left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right),\quad \boldsymbol{\nabla}\times\left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right),\quad \boldsymbol{\nabla}\times\left(\frac{\hat{\mathbf{z}}}{\rho}\right). \tag{2.142c}$$

Draw the vector fields. Visualize and interpret the action of the curl operator. Which of the above are curl free.

5. (**20 points.**) For studying a phenomenon on a plane it is convenient to breakup

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_\perp + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \tag{2.143}$$

$$\boldsymbol{\nabla}_\perp = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi}. \tag{2.144}$$

Verify the following identities:

$$\boldsymbol{\nabla}_\perp\cdot\left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 2\pi\delta^{(2)}(\boldsymbol{\rho}), \qquad \boldsymbol{\nabla}_\perp\times\left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 0, \tag{2.145a}$$

$$\boldsymbol{\nabla}_\perp\cdot\left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = 0, \qquad \boldsymbol{\nabla}_\perp\times\left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = \hat{\mathbf{z}}\,2\pi\delta^{(2)}(\boldsymbol{\rho}). \tag{2.145b}$$

6. (**30 points.**) The scale factors for cylindrical polar coordinates as

$$h_\rho = 1,\quad h_\phi = \rho,\quad h_z = 1. \tag{2.146}$$

The differential statement in rectangular coordinates is

$$d\mathbf{r} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}} \tag{2.147}$$

and the corresponding differential statement in cylindrical polar coordinates is

$$d\mathbf{r} = h_\rho d\rho\,\hat{\boldsymbol{\rho}} + h_\phi d\phi\,\hat{\boldsymbol{\phi}} + h_z dz\,\hat{\mathbf{z}}. \tag{2.148}$$

The gradient operator in rectangular coordinates is

$$\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \tag{2.149}$$

and in cylindrical polar coordinates it is

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{1}{h_\rho}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{h_\phi}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{1}{h_z}\frac{\partial}{\partial z}. \tag{2.150}$$

Let a vector field in rectangular coordinates

$$\mathbf{E} = \hat{\mathbf{i}}E_x(x,y,z) + \hat{\mathbf{j}}E_y(x,y,z) + \hat{\mathbf{k}}E_z(x,y,z) \tag{2.151}$$

be expressed in cylindrical polar coordinates as

$$\mathbf{E} = \hat{\boldsymbol{\rho}}E_\rho(\rho,\phi,z) + \hat{\boldsymbol{\phi}}E_\phi(\rho,\phi,z) + \hat{\mathbf{z}}E_z(\rho,\phi,z). \tag{2.152}$$

Show that

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{h_\rho h_\phi h_z} \left[\frac{\partial}{\partial \rho}(h_\phi h_z E_\rho) + \frac{\partial}{\partial \phi}(h_z h_\rho E_\phi) + \frac{\partial}{\partial z}(h_\rho h_\phi E_z) \right]. \tag{2.153}$$

Show that

$$\boldsymbol{\nabla} \times \mathbf{E} = \frac{1}{h_\rho h_\phi h_z} \begin{vmatrix} h_\rho \hat{\boldsymbol{\rho}} & h_\phi \hat{\boldsymbol{\phi}} & h_z \hat{\mathbf{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ h_\rho E_\rho & h_\phi E_\phi & h_z E_z \end{vmatrix}. \tag{2.154}$$

Show that

$$\nabla^2 = \frac{1}{h_\rho h_\phi h_z} \left[\frac{\partial}{\partial \rho} \frac{h_\phi h_z}{h_\rho} \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \phi} \frac{h_z h_\rho}{h_\phi} \frac{\partial}{\partial \phi} + \frac{\partial}{\partial z} \frac{h_\rho h_\phi}{h_z} \frac{\partial}{\partial z} \right]. \tag{2.155}$$

7. (**30 points.**) The scale factors for cylindrical polar coordinates are

$$h_\rho = 1, \quad h_\phi = \rho, \quad h_z = 1. \tag{2.156}$$

The gradient operator in rectangular coordinates is

$$\boldsymbol{\nabla} = \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \tag{2.157}$$

and in cylindrical polar coordinates it is

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}} \frac{1}{h_\rho} \frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}} \frac{1}{h_\phi} \frac{\partial}{\partial \phi} + \hat{\mathbf{z}} \frac{1}{h_z} \frac{\partial}{\partial z}. \tag{2.158}$$

Let a vector field in rectangular coordinates

$$\mathbf{B} = \hat{\mathbf{i}} E_x(x, y, z) + \hat{\mathbf{j}} E_y(x, y, z) + \hat{\mathbf{k}} E_z(x, y, z) \tag{2.159}$$

be expressed in cylindrical polar coordinates as

$$\mathbf{B} = \hat{\boldsymbol{\rho}} E_\rho(\rho, \phi, z) + \hat{\boldsymbol{\phi}} E_\phi(\rho, \phi, z) + \hat{\mathbf{z}} E_z(\rho, \phi, z). \tag{2.160}$$

The divergence operation is achieved using the relation

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \frac{1}{h_\rho h_\phi h_z} \left[\frac{\partial}{\partial \rho}(h_\phi h_z E_\rho) + \frac{\partial}{\partial \phi}(h_z h_\rho E_\phi) + \frac{\partial}{\partial z}(h_\rho h_\phi E_z) \right]. \tag{2.161}$$

and the curl operation is accomplished using

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{h_\rho h_\phi h_z} \begin{vmatrix} h_\rho \hat{\boldsymbol{\rho}} & h_\phi \hat{\boldsymbol{\phi}} & h_z \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ h_\rho E_\rho & h_\phi E_\phi & h_z E_z \end{vmatrix}. \tag{2.162}$$

Given

$$\mathbf{B} = \hat{\mathbf{z}} 2 \ln \frac{2L}{\rho}, \tag{2.163}$$

where L is a constant. Evaluate

$$\boldsymbol{\nabla} \cdot \mathbf{B} \quad \text{for} \quad \rho \neq 0 \tag{2.164}$$

and

$$\boldsymbol{\nabla} \times \mathbf{B} \quad \text{for} \quad \rho \neq 0. \tag{2.165}$$

2.5 Vector calculus in spherical polar coordinates

1. (**10 points.**) In spherical polar coordinates a point is coordinated by the intersection of family of spheres, cones, and half-planes, given by

$$r = \sqrt{x^2 + y^2 + z^2}, \tag{2.166a}$$

$$\theta = \tan^{-1} \sqrt{\frac{x^2 + y^2}{z^2}}, \tag{2.166b}$$

$$\phi = \tan^{-1} \frac{y}{x}, \tag{2.166c}$$

respectively. Show that the gradient of these surfaces are given by

$$\boldsymbol{\nabla} r = \hat{\mathbf{r}}, \qquad \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}, \tag{2.167a}$$

$$\boldsymbol{\nabla}\theta = \hat{\boldsymbol{\theta}}\,\frac{1}{r}, \qquad \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}, \tag{2.167b}$$

$$\boldsymbol{\nabla}\phi = \hat{\boldsymbol{\phi}}\,\frac{1}{r\sin\theta}, \qquad \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}, \tag{2.167c}$$

which are normal to the respective surfaces. Sketch the surfaces and the corresponding normal vectors. This illustrates that $\boldsymbol{\nabla}$(surface) is a vector (field) normal to the surface.

2. (**10 points.**) Using the gradient operator in spherical polar coordinates,

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}, \tag{2.168}$$

evaluate the following

$$\frac{\partial}{\partial r}\hat{\mathbf{r}} = 0, \qquad \frac{\partial}{\partial r}\hat{\boldsymbol{\theta}} = 0, \qquad \frac{\partial}{\partial r}\hat{\boldsymbol{\phi}} = 0, \tag{2.169a}$$

$$\frac{\partial}{\partial\theta}\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}, \qquad \frac{\partial}{\partial\theta}\hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}}, \qquad \frac{\partial}{\partial\theta}\hat{\boldsymbol{\phi}} = 0, \tag{2.169b}$$

$$\frac{\partial}{\partial\phi}\hat{\mathbf{r}} = \sin\theta\,\hat{\boldsymbol{\phi}}, \qquad \frac{\partial}{\partial\phi}\hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\boldsymbol{\phi}}, \qquad \frac{\partial}{\partial\phi}\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\rho}} = -(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}). \tag{2.169c}$$

Visualize the above variational statements graphically.

3. (**10 points.**) Evaluate the following divergence of vector fields.

$$\boldsymbol{\nabla}\cdot\hat{\mathbf{r}}, \quad \boldsymbol{\nabla}\cdot\hat{\boldsymbol{\theta}}, \quad \boldsymbol{\nabla}\cdot\hat{\boldsymbol{\phi}}, \tag{2.170a}$$

$$\boldsymbol{\nabla}\cdot(r^2\hat{\mathbf{r}}), \quad \boldsymbol{\nabla}\cdot(r^2\hat{\boldsymbol{\theta}}), \quad \boldsymbol{\nabla}\cdot(r^2\hat{\boldsymbol{\phi}}), \tag{2.170b}$$

$$\boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r}\right), \quad \boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\theta}}}{r}\right), \quad \boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\phi}}}{r}\right). \tag{2.170c}$$

Draw the vector fields. Visualize and interpret the action of the divergence operator. Which of the above are divergenceless.

4. (**10 points.**) Evaluate the following curl of vector fields.

$$\boldsymbol{\nabla}\times\hat{\mathbf{r}}, \quad \boldsymbol{\nabla}\times\hat{\boldsymbol{\theta}}, \quad \boldsymbol{\nabla}\times\hat{\boldsymbol{\phi}}, \tag{2.171a}$$

$$\boldsymbol{\nabla}\times(r^2\hat{\mathbf{r}}), \quad \boldsymbol{\nabla}\times(r^2\hat{\boldsymbol{\theta}}), \quad \boldsymbol{\nabla}\times(r^2\hat{\boldsymbol{\phi}}), \tag{2.171b}$$

$$\boldsymbol{\nabla}\times\left(\frac{\hat{\mathbf{r}}}{r}\right), \quad \boldsymbol{\nabla}\times\left(\frac{\hat{\boldsymbol{\theta}}}{r}\right), \quad \boldsymbol{\nabla}\times\left(\frac{\hat{\boldsymbol{\phi}}}{r}\right). \tag{2.171c}$$

Draw the vector fields. Visualize and interpret the action of the curl operator. Which of the above are curl free.

5. (**30 points.**) The scale factors for spherical polar coordinates as

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta. \tag{2.172}$$

The differential statement in rectangular coordinates is

$$d\mathbf{r} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}} \tag{2.173}$$

and the corresponding differential statement in spherical polar coordinates is

$$d\mathbf{r} = h_r dr\,\hat{\mathbf{r}} + h_\theta d\theta\,\hat{\boldsymbol{\theta}} + h_\phi d\phi\,\hat{\boldsymbol{\phi}}. \tag{2.174}$$

The gradient operator in rectangular coordinates is

$$\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \tag{2.175}$$

and in spherical polar coordinates it is

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{1}{h_r}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{h_\theta}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{h_\phi}\frac{\partial}{\partial \phi}. \tag{2.176}$$

Let a vector field in rectangular coordinates

$$\mathbf{E} = \hat{\mathbf{i}}E_x(x,y,z) + \hat{\mathbf{j}}E_y(x,y,z) + \hat{\mathbf{k}}E_z(x,y,z) \tag{2.177}$$

be expressed in spherical polar coordinates as

$$\mathbf{E} = \hat{\mathbf{r}}E_r(r,\theta,\phi) + \hat{\boldsymbol{\theta}}E_\theta(r,\theta,\phi) + \hat{\boldsymbol{\phi}}E_\phi(r,\theta,\phi). \tag{2.178}$$

Show that

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{h_r h_\theta h_\phi}\left[\frac{\partial}{\partial r}(h_\theta h_\phi E_r) + \frac{\partial}{\partial \theta}(h_\phi h_r E_\theta) + \frac{\partial}{\partial \phi}(h_r h_\theta E_\phi)\right]. \tag{2.179}$$

Show that

$$\boldsymbol{\nabla}\times\mathbf{E} = \frac{1}{h_r h_\theta h_\phi}\begin{vmatrix} h_r\hat{\mathbf{r}} & h_\theta\hat{\boldsymbol{\theta}} & h_\phi\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ h_r E_r & h_\theta E_\theta & h_\phi E_\phi \end{vmatrix}. \tag{2.180}$$

Show that

$$\nabla^2 = \frac{1}{h_r h_\theta h_\phi}\left[\frac{\partial}{\partial r}\frac{h_\theta h_\phi}{h_r}\frac{\partial}{\partial r} + \frac{\partial}{\partial \theta}\frac{h_\phi h_r}{h_\theta}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi}\frac{h_r h_\theta}{h_\phi}\frac{\partial}{\partial \phi}\right]. \tag{2.181}$$

2.6 Vector calculus on a surface

1. (**80 points.**) Let us consider the following fields that exist only the surface of a sphere of radius a:

$$\mathbf{E} = \hat{\boldsymbol{\theta}}\,\frac{\delta(r-a)}{2\pi r\sin\theta}, \tag{2.182a}$$

$$\mathbf{B} = \hat{\boldsymbol{\phi}}\,\frac{\delta(r-a)}{2\pi r\sin\theta}, \tag{2.182b}$$

where (r,θ,ϕ) are spherical polar coordinates.

(a) Illustrate the vector field lines for **E** and **B** using a diagram.

(b) Show that

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 0, \quad \theta \neq 0,\pi, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \quad \text{everywhere}, \tag{2.183a}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = 0, \quad \text{everywhere}, \qquad \boldsymbol{\nabla}\times\mathbf{B} = 0, \quad \theta \neq 0,\pi. \tag{2.183b}$$

(c) Further, using Gauss's theorem and Stoke's theorem, show that

$$\int_V d^3r\, \boldsymbol{\nabla}\cdot\mathbf{E} = \begin{cases} +1, & \text{if } V \text{ encloses } \theta = 0,\\ -1, & \text{if } V \text{ encloses } \theta = \pi, \end{cases} \tag{2.184a}$$

$$\int_S d\mathbf{a}\cdot\boldsymbol{\nabla}\times\mathbf{B} = \begin{cases} +1, & \text{if } S \text{ encloses } \theta = 0,\\ -1, & \text{if } S \text{ encloses } \theta = \pi, \end{cases} \tag{2.184b}$$

(2.184c)

where V represents the volume of a cone with apex at the origin with infinitely small opening angle, and S represents an infinitely small surface area on the sphere.

Thus, using the property of δ-function, deduce

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \left[\delta^{(3)}(\mathbf{r}-\mathbf{N}) - \delta^{(3)}(\mathbf{r}-\mathbf{S})\right], \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \tag{2.185a}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = 0, \qquad \boldsymbol{\nabla}\times\mathbf{B} = \hat{\mathbf{r}}\left[\delta^{(3)}(\mathbf{r}-\mathbf{N}) - \delta^{(3)}(\mathbf{r}-\mathbf{S})\right], \tag{2.185b}$$

where **N** represents the North pole and **S** represents the South pole on the sphere. In particular,

$$\delta^{(3)}(\mathbf{r}-\mathbf{N}) = \delta(r-a)\frac{\delta(\theta)\delta(\phi)}{r^2\sin\theta}, \tag{2.186a}$$

$$\delta^{(3)}(\mathbf{r}-\mathbf{S}) = \delta(r-a)\frac{\delta(\theta-\pi)\delta(\phi)}{r^2\sin\theta}. \tag{2.186b}$$

Comments:

(a) Thus, electromagnetism on a sphere will require charge densities to have $\rho(\pi-\theta,\phi+\pi) = -\rho(\theta,\phi)$. The suggestion seems to be that a positive charge on such a sphere will necessarily require there to be a negative charge on the diametrically opposite side. However, we could imagine a positive and negative charge that are not at diametrically opposite ends. Is this allowed?

(b) Think the development of spherical harmonics and the introduction of $y_\pm$ in Schwinger's lectures.

(c) Observe that

$$\boldsymbol{\nabla}_\perp \times \left(\frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\right) = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi, \tag{2.187}$$

which is naively expected to be zero.

2.7 Curvilinear coordinates (Outdated, written before Fall 2019)

1. (**80 points.**) A vector $\mathbf{v}$ in terms of the basis vectors $\mathbf{e}_i$ has the form

$$\mathbf{v} = \mathbf{e}_i v^i, \tag{2.188}$$

and in terms of another set of basis vectors $\bar{\mathbf{e}}_i$ has the form

$$\mathbf{v} = \bar{\mathbf{e}}_i \bar{v}^i. \tag{2.189}$$

If the two sets of basis vectors are related by the linear transformation

$$\bar{\mathbf{e}}_i = \mathbf{e}_j a^j{}_i, \tag{2.190}$$

then show that

$$\bar{v}^i = b^i{}_j v^j, \tag{2.191}$$

where $b = a^{-1}$.

(a) Spherical polar coordinates are defined using the transformations

$$x = r\sin\theta\cos\phi, \qquad r = \sqrt{x^2+y^2+z^2}, \tag{2.192a}$$

$$y = r\sin\theta\sin\phi, \qquad \theta = \tan^{-1}\sqrt{\frac{x^2+y^2}{z^2}}, \tag{2.192b}$$

$$z = r\cos\theta, \qquad \phi = \tan^{-1}\frac{y}{x}. \tag{2.192c}$$

Let us chose

$$\mathbf{e}_1 = \hat{\mathbf{i}}, \quad \mathbf{e}_2 = \hat{\mathbf{j}}, \quad \mathbf{e}_3 = \hat{\mathbf{k}}, \tag{2.193}$$

and

$$\bar{\mathbf{e}}_1 = \hat{\mathbf{r}} = \boldsymbol{\nabla} r, \quad \bar{\mathbf{e}}_2 = \boldsymbol{\theta} = \boldsymbol{\nabla}\theta, \quad \bar{\mathbf{e}}_3 = \boldsymbol{\phi} = \boldsymbol{\nabla}\phi. \tag{2.194}$$

Show that the linear transformation a connecting the two sets of basis vectors is

$$a = \begin{bmatrix} \sin\theta\cos\phi & \dfrac{\cos\theta\cos\phi}{r} & -\dfrac{\sin\phi}{r\sin\theta} \\ \sin\theta\sin\phi & \dfrac{\cos\theta\sin\phi}{r} & \dfrac{\cos\phi}{r\sin\theta} \\ \cos\theta & -\dfrac{\sin\theta}{r} & 0 \end{bmatrix} \tag{2.195}$$

and

$$b = a^{-1} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{bmatrix}. \tag{2.196}$$

(b) The differential of a position vector in these basis set takes the form

$$d\mathbf{x} = \hat{\mathbf{i}}dx + \hat{\mathbf{j}}dy + \hat{\mathbf{k}}dz = \hat{\mathbf{r}}d\bar{x} + \boldsymbol{\theta}d\bar{y} + \boldsymbol{\phi}d\bar{z}. \tag{2.197}$$

Using Eq. (2.190) we learn that

$$\boldsymbol{\theta} = \frac{\hat{\boldsymbol{\theta}}}{r}, \quad \boldsymbol{\phi} = \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}, \tag{2.198}$$

where

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}, \tag{2.199a}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}, \tag{2.199b}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}, \tag{2.199c}$$

Using Eq. (2.191) and replacing total differential for the sum of partial differentials we learn that

$$d\bar{x} = \sin\theta\cos\phi\,dx + \sin\theta\sin\phi\,dy + \cos\theta\,dz = dr, \tag{2.200a}$$

$$d\bar{y} = r\cos\theta\cos\phi\,dx + r\cos\theta\sin\phi\,dy - r\sin\theta\,dz = r^2 d\theta, \tag{2.200b}$$

$$d\bar{z} = -r\sin\theta\sin\phi\,dx + r\sin\theta\cos\phi\,dy = r^2\sin^2\theta d\phi. \tag{2.200c}$$

Thus, show that

$$d\mathbf{x} = \hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\theta}}\,rd\theta + \hat{\boldsymbol{\phi}}\,r\sin\theta d\phi. \tag{2.201}$$

The scale factors are read out to be

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta. \tag{2.202}$$

(c) The metric tensor is defined using the inner product,

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j, \tag{2.203a}$$

$$\bar{g}_{ij} = \bar{\mathbf{e}}_i \cdot \bar{\mathbf{e}}_j. \tag{2.203b}$$

Show that

$$g_{ij} = \delta_{ij} \tag{2.204}$$

and

$$\bar{g}_{ij} = g_{mn} a^m{}_i a^n{}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{r^2} & 0 \\ 0 & 0 & \dfrac{1}{r^2 \sin^2\theta} \end{bmatrix}. \tag{2.205}$$

Further, show that

$$d\mathbf{x} \cdot d\mathbf{x} = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{2.206}$$

2. (**80 points.**) Cylindrical polar coordinates are defined by the transformations

$$x = \rho\cos\phi, \qquad \rho = \sqrt{x^2 + y^2}, \tag{2.207a}$$

$$y = \rho\sin\phi, \qquad \phi = \tan^{-1}\frac{y}{x}, \tag{2.207b}$$

$$z = z, \qquad z = z. \tag{2.207c}$$

Cylindrical polar coordinates form a orthogonal curvilinear coordinate system. List the three family of surfaces represented by these equations. Also, list the corresponding lines of flow (normal to the these surfaces) that serve as the coordinate lines.

(a) Using the differential statement

$$dx = \frac{\partial x}{\partial \rho} d\rho + \frac{\partial x}{\partial \phi} d\phi + \frac{\partial x}{\partial z} dz \tag{2.208}$$

and similar relations for dy and dz show that

$$dx^2 + dy^2 + dz^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2. \tag{2.209}$$

Thus read out the scale factors for cylindrical polar coordinates as

$$h_\rho = 1, \quad h_\phi = \rho, \quad h_z = 1. \tag{2.210}$$

Let

$$\mathbf{R} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.211}$$

Show that

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \mathbf{R} \begin{pmatrix} h_\rho d\rho \\ h_\rho d\phi \\ h_z dz \end{pmatrix}. \tag{2.212}$$

(b) Similarly, using the differential statement

$$\frac{\partial}{\partial x} = \frac{\partial \rho}{\partial x}\frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} \tag{2.213}$$

and similar relations for $\partial/\partial y$ and $\partial/\partial z$ show that

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \mathbf{R} \begin{pmatrix} \frac{1}{h_\rho}\frac{\partial}{\partial \rho} \\ \frac{1}{h_\phi}\frac{\partial}{\partial \phi} \\ \frac{1}{h_z}\frac{\partial}{\partial z} \end{pmatrix}. \tag{2.214}$$

(c) Using the construction of the unit vector

$$\hat{\boldsymbol{\rho}} = \frac{\boldsymbol{\nabla}\rho}{|\boldsymbol{\nabla}\rho|} \tag{2.215}$$

and similar constructions for $\hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{z}}$ show that

$$\begin{pmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{pmatrix} = \mathbf{R} \begin{pmatrix} \hat{\boldsymbol{\rho}} \\ \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} \end{pmatrix}. \tag{2.216}$$

(d) Starting from the differential statement in rectangular coordinates

$$d\mathbf{r} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}} \tag{2.217}$$

derive the corresponding differential statement in cylindrical polar coordinates

$$d\mathbf{r} = h_\rho d\rho\,\hat{\boldsymbol{\rho}} + h_\phi d\phi\,\hat{\boldsymbol{\phi}} + h_z dz\,\hat{\mathbf{z}}. \tag{2.218}$$

(e) Using the expression for the gradient operator in rectangular coordinates

$$\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \tag{2.219}$$

derive the expression for the gradient operator in cylindrical polar coordinates

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{1}{h_\rho}\frac{\partial}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{h_\phi}\frac{\partial}{\partial \phi} + \hat{\mathbf{z}}\frac{1}{h_z}\frac{\partial}{\partial z}. \tag{2.220}$$

(f) Let a function $f(x,y,z)$ in cylindrical polar coordinates be $\bar{f}(\rho,\phi,z)$. That is, $f(x,y,z) = \bar{f}(\rho,\phi,z)$. Show that

$$df = d\mathbf{r}\cdot\boldsymbol{\nabla} f = \left[dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right] f(x,y,z). \tag{2.221}$$

Show that

$$df = \left[d\rho\frac{\partial}{\partial \rho} + d\phi\frac{\partial}{\partial \phi} + dz\frac{\partial}{\partial z}\right] \bar{f}(\rho,\phi,z) = d\bar{f}. \tag{2.222}$$

This is the definition of a scalar field.

(g) Consider the vector field in rectangular coordinates

$$\mathbf{E} = \hat{\mathbf{i}}E_x(x,y,z) + \hat{\mathbf{j}}E_y(x,y,z) + \hat{\mathbf{k}}E_z(x,y,z). \tag{2.223}$$

A vector field, by definition, in cylindrical coordinates is given by

$$\begin{pmatrix} E_x(x,y,z) \\ E_y(x,y,z) \\ E_z(x,y,z) \end{pmatrix} = \mathbf{R} \begin{pmatrix} E_\rho(\rho,\phi,z) \\ E_\phi(\rho,\phi,z) \\ E_z(\rho,\phi,z) \end{pmatrix}. \tag{2.224}$$

Thus, show that

$$\mathbf{E} = \hat{\boldsymbol{\rho}}E_\rho(\rho,\phi,z) + \hat{\boldsymbol{\phi}}E_\phi(\rho,\phi,z) + \hat{\mathbf{z}}E_z(\rho,\phi,z). \tag{2.225}$$

(h) Derive the following nine derivatives

$$\begin{pmatrix} \frac{\partial\hat{\boldsymbol{\rho}}}{\partial\rho} & \frac{\partial\hat{\boldsymbol{\phi}}}{\partial\rho} & \frac{\partial\hat{\mathbf{z}}}{\partial\rho} \\ \frac{\partial\hat{\boldsymbol{\rho}}}{\partial\phi} & \frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} & \frac{\partial\hat{\mathbf{z}}}{\partial\phi} \\ \frac{\partial\hat{\boldsymbol{\rho}}}{\partial z} & \frac{\partial\hat{\boldsymbol{\phi}}}{\partial z} & \frac{\partial\hat{\mathbf{z}}}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \hat{\boldsymbol{\phi}} & -\hat{\boldsymbol{\rho}} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.226}$$

(i) Show that

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{h_\rho h_\phi h_z} \left[\frac{\partial}{\partial \rho}(h_\phi h_z E_\rho) + \frac{\partial}{\partial \phi}(h_z h_\rho E_\phi) + \frac{\partial}{\partial z}(h_\rho h_\phi E_z) \right]. \tag{2.227}$$

(j) Show that

$$\boldsymbol{\nabla} \times \mathbf{E} = \frac{1}{h_\rho h_\phi h_z} \begin{vmatrix} h_\rho \hat{\boldsymbol{\rho}} & h_\phi \hat{\boldsymbol{\phi}} & h_z \hat{\mathbf{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ h_\rho E_\rho & h_\phi E_\phi & h_z E_z \end{vmatrix}. \tag{2.228}$$

(k) Show that

$$\nabla^2 = \frac{1}{h_\rho h_\phi h_z} \left[\frac{\partial}{\partial \rho} \frac{h_\phi h_z}{h_\rho} \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \phi} \frac{h_z h_\rho}{h_\phi} \frac{\partial}{\partial \phi} + \frac{\partial}{\partial z} \frac{h_\rho h_\phi}{h_z} \frac{\partial}{\partial z} \right]. \tag{2.229}$$

Chapter 3

Functions of a complex variable

3.1 Complex number

3.1.1 Addition, subtraction, multiplication, division, and exponentiation

1. (**10 points.**) Find the real and imaginary part of

$$z = \frac{(a+ib)}{(c+id)}. \tag{3.1}$$

Thus, express z in the form $z = u + iv$. Assume a, b, c, and d are real.

2. (**10 points.**) Find the real and imaginary part of the following functions of the complex variable $z = x+iy$ in terms of x and y.

$$f = \frac{1}{z}, \tag{3.2a}$$

$$f = \frac{1}{z^2}. \tag{3.2b}$$

3.1.2 Polar representation (Argand diagram): Addition, subtraction, multiplication, division, and exponentiation

1. (**Polar representation.**) Polar representation of a complex number

$$z = x + iy \tag{3.3}$$

in terms of the magnitude r and the phase ϕ,

$$x = r\cos\phi, \qquad r = \sqrt{x^2+y^2}, \tag{3.4a}$$

$$y = r\sin\phi, \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right), \tag{3.4b}$$

is

$$z = r(\cos\phi + i\sin\phi). \tag{3.5}$$

Using the Euler formula

$$e^{i\phi} = \cos\phi + i\sin\phi, \tag{3.6}$$

which allows us to write a complex number in the form

$$z = re^{i\phi}. \tag{3.7}$$

2. (**Euler formula.**) Recall the power series representation of the exponential function, the cosine function, and the sine function,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \tag{3.8a}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, \tag{3.8b}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, \tag{3.8c}$$

which is typically introduced for real x. However, the power series representation is valid for complex variables too. For the phase of a complex number ϕ, which is real, using the series representation, verify that

$$e^{i\phi} = \cos\phi + i\sin\phi, \tag{3.9}$$

This is called the Euler formula.

3. (**Example.**) For two complex numbers,

$$z_1 = r_1(\cos\phi_1 + i\sin\phi_1) = r_1 e^{i\phi_1}, \tag{3.10a}$$

$$z_2 = r_2(\cos\phi_2 + i\sin\phi_2) = r_2 e^{i\phi_2}, \tag{3.10b}$$

verify that

$$z_1 z_2 = r_1 r_2 \Big[\cos(\phi_1 + \phi_2) + i\sin(\phi_1 + \phi_2)\Big] = r_1 r_2 e^{i(\phi_1+\phi_2)}. \tag{3.11}$$

4. (**DeMoivre's theorem.**) Show using trignometric identities, (without relying on the Euler formula,) that

$$(\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi. \tag{3.12}$$

This is called the DeMoivre theorem. The statement of DeMoivre's theorem is immediate using the Euler formula.

5. (**Example.**) Find the real and imaginary part of the following functions of the complex variable $z = re^{i\phi}$ in terms of the magnitude r and the phase ϕ. Thus, verify that

$$z^3 = r^3\cos 3\phi + ir^3\sin 3\phi, \tag{3.13a}$$

$$\frac{1}{z^3} = \frac{\cos 3\phi}{r^3} - i\frac{\sin 3\phi}{r^3}, \tag{3.13b}$$

$$\sqrt[3]{z} = r^{\frac{1}{3}}\cos\left(\frac{\phi}{3}\right) + ir^{\frac{1}{3}}\sin\left(\frac{\phi}{3}\right). \tag{3.13c}$$

Actually, the cube root $\sqrt[3]{z}$ leads to three independent solutions, the above being one of the three.

6. (**10 points.**) For a given complex number z, say

$$z = \sqrt{2}\, e^{i\frac{\pi}{3}}, \tag{3.14}$$

evaluate

$$z^2, z^3, z^4, z^5, z^6, z^7, z^8, z^9, z^{10}. \tag{3.15}$$

Mark all of them on the complex plane. Decipher the pattern.

7. (**20 points.**) Verify that

$$\sqrt{-2}\sqrt{-3} = -\sqrt{6}. \tag{3.16}$$

However, it is often tempting to conclude

$$\sqrt{-2}\sqrt{-3} = \sqrt{(-2)(-3)} = \sqrt{6}. \tag{3.17}$$

The ambiguity in the interpretation of $\sqrt{-2}$ and $\sqrt{-3}$ is (partly) removed by writing,

$$\sqrt{-2} = \left(2e^{i\pi}\right)^{\frac{1}{2}} = \sqrt{2}e^{i\frac{\pi}{2}}, \tag{3.18a}$$

$$\sqrt{-3} = \left(3e^{i\pi}\right)^{\frac{1}{2}} = \sqrt{3}e^{i\frac{\pi}{2}}. \tag{3.18b}$$

This only partly removes the ambiguity because $\sqrt{-2}$ and $\sqrt{-3}$ have two independent roots each and Eqs. (3.18) only identifies one of the roots, the principal root, for each. Using Eqs. (3.18) verify the correctness of the statement in Eq. (3.16) again. The above ambiguity in the interpretation and the related confusions plagued the development of ideas related to complex numbers until the geometric visualization of a complex number using Argand diagram (magnitude and direction in polar representation) was discovered by Wessel in 1797 and popularized by Argand in 1806. Without this geometric interpretation even Euler fell into the trap of concluding $\sqrt{-2}\sqrt{-3} = \sqrt{6}$. So, is the statement in Eq. (3.17) erroneous? No. To this end, let us remove the ambiguity completely by recognizing the multiplicities in the roots,

$$\sqrt{-2} = \left(2e^{i\pi}\right)^{\frac{1}{2}} = \sqrt{2}e^{i\frac{\pi}{2}}(1,\omega), \quad \omega = e^{i\pi}, \tag{3.19a}$$

$$\sqrt{-3} = \left(3e^{i\pi}\right)^{\frac{1}{2}} = \sqrt{3}e^{i\frac{\pi}{2}}(1,\omega), \quad \omega = e^{i\pi}, \tag{3.19b}$$

where comma-separated quantities contribute to multiplicities in roots. Muliplication of the two roots of $\sqrt{-2}$ and two roots of $\sqrt{-3}$ leads to four possibilities,

$$(1,\omega)\times(1,\omega) \to (1,\omega,\omega,\omega^2). \tag{3.20}$$

Using $\omega^2 = 1$, only two out of four possibilities are independent. Thus, we have

$$\sqrt{-2}\sqrt{-3} = \sqrt{2}\sqrt{3}(1,\omega), \quad \omega = e^{i\pi}. \tag{3.21}$$

In summary, both the statements in Eqs. (3.16) and (3.17) are correct.

8. (**20 points.**) Evaluate

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{23}. \tag{3.22}$$

Mark the resulting number on the complex plane.

9. (**20 points.**) Prove the identity

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}. \tag{3.23}$$

Use the identity

$$(2+i)(3+i) = 5 + i5. \tag{3.24}$$

Similarly, find y/x in the relation

$$\tan^{-1}\left(\frac{3}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{y}{x}\right). \tag{3.25}$$

Solution: $y/x = 17/7$.

10. (**20 points.**) Find the real and imaginary part of the function

$$f = \cos z = \frac{e^{iz} + e^{-iz}}{2} \tag{3.26}$$

in terms of x and y, where $z = x + iy$ is a complex variable.

11. (**Example.**) Find the real and imaginary part of the following functions. Thus, verify that

$$e^{iz} = e^{-y}(\cos x + i \sin x), \tag{3.27a}$$
$$\ln z = \ln r + i(\phi + 2\pi n), \tag{3.27b}$$

where n is an arbitrary integer.

12. (**20 points.**) Find the real and imaginary part of the function

$$f = \sqrt{z}. \tag{3.28}$$

13. (**10 points.**) (Refer Arfken) The complex quantities

$$a = u + iv, \tag{3.29a}$$
$$b = x + iy \tag{3.29b}$$

may also be represented as two-dimensional vectors

$$\mathbf{a} = \hat{\mathbf{x}}\, u + \hat{\mathbf{y}}\, v, \tag{3.30a}$$
$$\mathbf{b} = \hat{\mathbf{x}}\, x + \hat{\mathbf{y}}\, y. \tag{3.30b}$$

Show that

$$(a^*)b = \mathbf{a} \cdot \mathbf{b} + i\hat{\mathbf{z}} \cdot \mathbf{a} \times \mathbf{b}. \tag{3.31}$$

14. (**20 points.**) The close connection between the geometry of a complex number

$$z = x + iy \tag{3.32}$$

and a two-dimensional vector

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} \tag{3.33}$$

is intriguing. They have the same rules for addition and subtraction, but differ in their rules for multiplication. Show that

$$z_1^* z_2 = (\mathbf{r}_1 \cdot \mathbf{r}_2) + i(\mathbf{r}_1 \times \mathbf{r}_2) \cdot \hat{\mathbf{k}}. \tag{3.34}$$

In the quest for a number system that corresponds to a three dimensional vector, Hamilton in 1843 invented the quaternions. A quaternion P can be expressed in terms of Pauli matrices as

$$P = a_0 - i\mathbf{a} \cdot \boldsymbol{\sigma}. \tag{3.35}$$

Recall that the Pauli matrices are completely characterized by the identity

$$(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) = (\mathbf{a} \cdot \mathbf{b}) + i(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{3.36}$$

(a) Show that the (Hamilton) product of two quaternions,

$$P = a_0 - i\mathbf{a} \cdot \boldsymbol{\sigma}, \tag{3.37a}$$
$$Q = b_0 - i\mathbf{b} \cdot \boldsymbol{\sigma}, \tag{3.37b}$$

is given by

$$PQ = (a_0 b_0 - \mathbf{a} \cdot \mathbf{b}) - i(a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{3.38}$$

(b) Verify that the Hamilton product is non-commutative. Determine $\big[P, Q\big]$.
Solution:

$$\big[P, Q\big] = -2i(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}. \tag{3.39}$$

15. (**20 points.**) Given

$$z_1 = e^{i\pi\frac{5}{4}}, \tag{3.40}$$
$$z_2 = e^{i\pi\frac{3}{2}}. \tag{3.41}$$

Show that

$$z_1 z_2 = e^{i\pi\frac{3}{4}}. \tag{3.42}$$

Mark z_1, z_2, and $z_1 z_2$, on the complex plane.

16. (**20 points.**) Given

$$z_1 = 3i - 2, \tag{3.43}$$
$$z_2 = 3 + 2i. \tag{3.44}$$

Evaluate

$$\frac{z_1}{z_2}. \tag{3.45}$$

Mark z_1, z_2, and z_1/z_2, on the complex plane. Decipher the pattern in this construction and find this class of numbers. **Solution:** $z_2 = re^{i\theta}$ and $z_1 = iz_2$.

3.1.3 Cardano formula

1. (**20 points.**) (NOT COMPLETE) The roots to the cubic equation

$$x^3 + px + q = 0 \tag{3.46}$$

can be found using the Cardano formula

$$x = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}}. \tag{3.47}$$

For $p = -15$ and $q = -4$, verify by substitution that each of

$$4, \quad -2 \pm \sqrt{3}. \tag{3.48}$$

are roots. Using Cardano formula we have

$$x = (2 + i11)^{\frac{1}{3}} + (2 - i11)^{\frac{1}{3}}. \tag{3.49}$$

The three cube roots for the two terms are

$$(2 + i11)^{\frac{1}{3}} = z(1, \omega, \omega^*), \qquad z = 2 + i, \tag{3.50}$$
$$(2 - i11)^{\frac{1}{3}} = z^*(1, \omega^*, \omega), \tag{3.51}$$

This has the form

$$x = z(1, \omega, \omega^*) + z^*(1, \omega^*, \omega), \tag{3.52}$$

and allows 9 combinations,

$$\begin{matrix} z + z^* & z + z^*\omega^* & z + z^*\omega \\ \left[z + z^*\omega^*\right]^* & z\omega + (z\omega)^* & z\omega + z^*\omega \\ \left[z + z^*\omega\right]^* & \left[z\omega + z^*\omega\right]^* & z\omega^* + (z\omega^*)^* \end{matrix} \tag{3.53}$$

The diagonal combinations are real.

3.1.4 Roots of unity

1. (**20 points.**) Find the two roots that satisfy the equation

$$z^2 = i. \tag{3.54}$$

   Mark the points corresponding to the two roots on the complex plane.
   Solution: The two roots are

$$e^{i\frac{\pi}{4}}(1, \omega), \qquad \omega = e^{i\pi} = -1. \tag{3.55}$$

2. (**20 points.**) Find the two roots that satisfy the equation

$$z^2 = -i. \tag{3.56}$$

   Mark the points corresponding to the two roots on the complex plane.
   Solution: The two roots are

$$e^{i\frac{3\pi}{4}}(1, \omega), \qquad \omega = e^{i\pi} = -1, \tag{3.57}$$

   or

$$e^{-i\frac{\pi}{4}}(1, \omega), \qquad \omega = e^{i\pi} = -1, \tag{3.58}$$

3. (**20 points.**) Find the cube roots of unity by solving the equation

$$z^3 = 1. \tag{3.59}$$

   Mark the points corresponding to the three roots on the complex plane.

4. (**20 points.**) Find the three roots that satisfy the equation

$$z^3 = i. \tag{3.60}$$

   Mark the points corresponding to the three roots on the complex plane.
   Solution: The three roots are

$$e^{i\frac{\pi}{6}}(1, \omega, \omega^2), \qquad \omega = e^{i\frac{2\pi}{3}}. \tag{3.61}$$

5. (**20 points.**) Find the four roots that satisfy the equation

$$z^4 = -1. \tag{3.62}$$

   Mark the points corresponding to the three roots on the complex plane.
   Solution: The four roots are

$$e^{i\frac{\pi}{4}}(1, \omega, \omega^2, \omega^3), \qquad \omega = e^{i\frac{\pi}{2}}. \tag{3.63}$$

6. (**30 points.**) Find the fifth roots of unity by solving the equation

$$z^5 = 1. \tag{3.64}$$

   Mark the points corresponding to the five roots on the complex plane. Find the five roots of the equation

$$z^5 = -1. \tag{3.65}$$

   Mark the roots on the complex plane. Next, find the roots of the equation

$$z^5 = i \tag{3.66}$$

   and mark the roots on the complex plane. Repeat the exercise for $z^5 = -i$. How do these roots match with the fifth roots of unity? Recognize the pattern.

7. (**20 points.**) Find the three roots of -1 by solving the equation

$$z^3 = -1. \tag{3.67}$$

Mark the the points corresponding to the three roots on the complex plane.

8. (**20 points.**) Find the cube roots of unity by solving the equation

$$z^3 = 1, \tag{3.68}$$

where the exponent of z is a positive integer.

   (a) Find the roots of the equation

$$z^{\frac{3}{2}} = 1, \tag{3.69}$$

   where the exponent of z is a rational number.

   (b) Find the roots of the equation

$$z^{\pi} = 1, \tag{3.70}$$

   where the exponent of z is an irrational number.

9. (**20 points.**) Find all z that satisfies the equation

$$e^z = e^{iz}. \tag{3.71}$$

Show them on a complex plane.

10. (**20 points.**) Find all z that satisfies the following and show them on a complex plane.

$$\begin{aligned} e^z &= e^{ia}, && (3.72a)\\ e^z &= e^{a}, && (3.72b)\\ e^z &= e^{iz}, && (3.72c)\\ e^{i2\pi z} &= 1, && (3.72d)\\ e^{2\pi z} &= 1, && (3.72e)\\ e^{i2\pi z^2} &= 1, && (3.72f)\\ e^{i2\pi z^2} &= e^{i4\pi z}. && (3.72g) \end{aligned}$$

Solution: a) $z = ia + i2\pi n$, b) $z = a + i2\pi n$, c) $z = (-1+i)\pi n$, d) $z = n$, e) $z = in$, f) $z = \sqrt{n}$, g) $z = 1 \pm \sqrt{1+n}$.

11. (**20 points.**) Locate $z = \pi^i$ on the complex plane.

12. (**20 points.**) Locate $z = i^i$ on the complex plane.

13. (**20 points.**) Show that the product of cube roots of unity is equal to -1, if $n = 2$, and equal to 1, if $n = 2, 3, 4, \ldots$.

14. (**20 points.**) (INCOMLETE) Show that

$$z_1^{\frac{1}{n_1}} z_2^{\frac{1}{n_2}} = r_1^{\frac{1}{n_1}} r_2^{\frac{1}{n_2}} e^{i\left(\frac{\theta_1}{n_1} + \frac{\theta_2}{n_2}\right)} e^{i\frac{2\pi}{n_1 n_2}(k_1 n_2 + k_2 n_1)}, \tag{3.73}$$

where $k_1 = 0, 1, 2, \ldots, n_1 - 1$ and $k_2 = 0, 1, 2, \ldots, n_2 - 1$. The multiplicities are determined by

$$k = (k_1 n_2 + k_2 n_1) \bmod n_1 n_2. \tag{3.74}$$

n_1	n_2	k
2	3	0,1,2,...,5
2×2	3	0,1,2,...,11
2×2	3×2	0,2,4,...,22
3×2	3×2	0,6,12,...,30
2×2	3×3	0,1,2,...,35

Recognize the generic pattern.

3.1.5 Hyperbolic functions

1. (**100 points.**) Hyperbolic cosine function and sine function are defined using the exponential function. We have

$$\cosh x = \frac{e^x + e^{-x}}{2}, \tag{3.75a}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}. \tag{3.75b}$$

Here, and in the following, assume x and y to be real. Recall that the corresponding trigonometric functions are defined in terms of the exponential function as

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \tag{3.76a}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{3.76b}$$

Hyperbolic functions extend the domain of the corresponding trigonometric functions to the complex plane.

(a) Show that

$$\cosh x = \cos(ix), \tag{3.77a}$$

$$\sinh x = -i\sin(ix). \tag{3.77b}$$

(b) Plot $\cosh x$ and $\sinh x$ as functions of x.

(c) Using Eqs. (3.75) derive the identity

$$\cosh^2 x - \sinh^2 x = 1. \tag{3.78}$$

Derive the identities for the sum of arguments of hyperbolic functions,

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y, \tag{3.79a}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y. \tag{3.79b}$$

Derive the derivative operations on hyperbolic functions,

$$\frac{d}{dx}\cosh x = \sinh x, \tag{3.80a}$$

$$\frac{d}{dx}\sinh x = \cosh x, \tag{3.80b}$$

and the integral operations,

$$\int dx\, \cosh x = \sinh x, \tag{3.81a}$$

$$\int dx\, \sinh x = \cosh x. \tag{3.81b}$$

(d) To find the inverse hyperbolic function of sine let us define

$$y = \sinh^{-1} x. \tag{3.82}$$

Then, we have

$$x = \sinh y = \frac{e^y - e^{-y}}{2}, \tag{3.83}$$

which can be rewritten as a quadratic equation in e^y,

$$(e^y)^2 - 2x(e^y) - 1 = 0, \tag{3.84}$$

with solutions

$$e^y = x \pm \sqrt{x^2 + 1}. \tag{3.85}$$

Presuming y to be real argue that only one of the roots is consistent with $e^y > 0$. Taking logarithm we have

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}). \tag{3.86}$$

Similarly, derive the expression

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}). \tag{3.87}$$

2. (**20 points.**) Hyperbolic cosine and sine are defined in terms of the exponential function,

$$\cosh x = \frac{e^x + e^{-x}}{2}, \tag{3.88a}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}. \tag{3.88b}$$

Using the above prove the identity

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y. \tag{3.89}$$

3.2 Cauchy-Riemann conditions

1. (**20 points.**) Recall that analytic functions satisfy the Cauchy-Riemann conditions. That is, the real and imaginary parts of an analytic function

$$f(x + iy) = u(x, y) + iv(x, y) \tag{3.90}$$

satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{3.91a}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{3.91b}$$

In terms of the variables z and z^*, defined using

$$z = x + iy, \qquad x = \frac{z + z^*}{2}, \tag{3.92a}$$

$$z^* = x - iy, \qquad y = \frac{z - z^*}{2i}, \tag{3.92b}$$

we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial z^*}, \qquad \frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}, \tag{3.93a}$$

$$\frac{\partial}{\partial y} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial z^*}, \qquad \frac{\partial}{\partial z^*} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}. \tag{3.93b}$$

(a) Show that the conditions in Eqs. (3.91) imply

$$\frac{\partial f}{\partial z^*} = -\frac{\partial}{\partial z}(u - iv), \tag{3.94a}$$

$$\frac{\partial f}{\partial z^*} = +\frac{\partial}{\partial z}(u - iv), \tag{3.94b}$$

respectively. Thus, show that the conditions in Eqs. (3.91) imply

$$\frac{\partial f}{\partial z^*} = 0, \tag{3.95}$$

which is insightful.

(b) Is the inverse true? That is, does the condition in Eq. (3.95) imply the conditions in Eqs. (3.91). To this end, begin from Eq. (3.95) and immediately conclude

$$\frac{\partial u}{\partial z^*} + i\frac{\partial v}{\partial z^*} = 0. \tag{3.96}$$

Then, proceed to derive

$$\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0, \tag{3.97}$$

which implies the conditions in Eqs. (3.91).

2. (**20 points.**) Show that the sums, products, and composites of analytic functions are also analytic.

3. (**20 points.**) Show that if $f(z)$ is an analytic function then the derivative df/dz satisfies the Cauchy-Riemann equations.

4. (**20 points.**) Analytic functions satisfy the Cauchy-Riemann equations. That is, the real and imaginary parts of an analytic function

$$f(x + iy) = u(x, y) + iv(x, y) \tag{3.98}$$

satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{3.99a}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{3.99b}$$

Given $f(z)$ and $g(z)$ are analytic functions in a region, then show that $f(g(z))$ satisfies the Cauchy-Riemann equations there.

Hint: Let $g = u + iv$ and $f = U + iV$. Thus, we can write

$$f(g(z)) = U(u(x, y), v(x, y)) + iV(u(x, y), v(x, y)). \tag{3.100}$$

5. (**20 points.**) Analytic functions satisfy the Cauchy-Riemann equations. That is, the real and imaginary parts of an analytic function

$$f(x + iy) = u(x, y) + iv(x, y) \tag{3.101}$$

satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{3.102a}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{3.102b}$$

Given $f(z)$ is an analytic function in a region, then show that $f^{-1}(z)$ satisfies the Cauchy-Riemann equations there.

Hint: Using

$$f^{-1}(f(z)) = z \tag{3.103}$$

we have

$$x = x(u(x,y), v(x,y)), \tag{3.104a}$$
$$y = y(u(x,y), v(x,y)). \tag{3.104b}$$

Thus, we have

$$1 = \frac{\partial x}{\partial x} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x}, \tag{3.105a}$$
$$0 = \frac{\partial x}{\partial y} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial y}, \tag{3.105b}$$
$$0 = \frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x}, \tag{3.105c}$$
$$1 = \frac{\partial y}{\partial y} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial y}, \tag{3.105d}$$

which are contained in

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.106}$$

Taking the inverse we obtain the relation

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y} & -\frac{\partial u}{\partial y} \\ -\frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix} \frac{1}{\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right]}. \tag{3.107}$$

Since $u(x,y)$ and $v(x,y)$ satisfy Cauchy-Riemann conditions the matrix relation takes the form

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix} \frac{1}{\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]}. \tag{3.108}$$

Thus, conclude

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}, \tag{3.109a}$$
$$\frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}. \tag{3.109b}$$

6. (**20 points.**) Express the Cauchy-Riemann equations in polar form:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \tag{3.110a}$$
$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}. \tag{3.110b}$$

7. (**20 points.**) Investigate if the function
$$f = \frac{z}{z^*} \tag{3.111}$$
is locally isotropic around the point $z = 0$. In particular, inquire the following:
$$\lim_{x\to 0}\lim_{y\to 0} f, \tag{3.112a}$$
$$\lim_{y\to 0}\lim_{x\to 0} f. \tag{3.112b}$$
Are they equal? Interpret the direction of approach in each of the above limits.

8. (**20 points.**) Let
$$u(x,y) = \frac{x^3 - y^3}{x^2 + y^2}. \tag{3.113}$$
Then evaluate the following:
$$\lim_{x\to 0}\lim_{y\to 0} \frac{\partial u}{\partial x}, \tag{3.114a}$$
$$\lim_{y\to 0}\lim_{x\to 0} \frac{\partial u}{\partial x}. \tag{3.114b}$$
Are they equal?

9. (**20 points.**) Verify that a real function f having dependence on the spatial variables (x, y, z) of the form $f(ax + by + cz)$ satisfies the Laplace equation in three dimensions
$$\nabla^2 f = 0 \tag{3.115}$$
provided
$$a^2 + b^2 + c^2 = 0. \tag{3.116}$$
Thus, deduce that $f(x + iy)$ satisfies the Laplace equation in two dimensions, because $1^2 + i^2 = 0$.

10. (**20 points.**) Given an analytic function
$$f(z) = u(x,y) + iv(x,y) \tag{3.117}$$
on a complex plane, $z = x + iy$, we can imagine the two functions $u(x,y)$ and $v(x,y)$ to exist on the two-dimensional plane spanned by the real variables x and y. (Recall that even though addition and subtraction are identical in these spaces, the algebra of multiplication is different. Division is not introduced in a vector space.) In terms of the gradient operator in the two-dimensional vector space,
$$\boldsymbol{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y}, \tag{3.118}$$
show that Cauchy-Riemann equations for $u(x,y)$ and $v(x,y)$ imply
$$(\boldsymbol{\nabla} u)\cdot(\boldsymbol{\nabla} v) = 0. \tag{3.119}$$
Thus, interpret that u's and v's are orthogonal family of surfaces and thus serves as a suitable chart for coordinatization. Mathematica allows visualization of these surfaces using the command ComplexContourPlot.

```
f[z_] = z^3;
ComplexContourPlot[ReIm[f[z]], {z,-3-3 I,3+3 I}]
```

The above two-line code in Mathematica plots the real and imaginary surfaces associated with the analytic function $f(z) = z^3$ between the coordinate points $(-3, -3)$ and $(3, 3)$.

11. (**20 points.**) We shall show that transformations governing an analytic function

$$f(z) = f(x + iy) = u(x, y) + iv(x, y) \tag{3.120}$$

are conformal. That is, they preserve angles. Using the differentials at z

$$dz_1 = |dz_1|e^{i\theta_1}, \qquad dz_2 = |dz_2|e^{i\theta_2}, \tag{3.121}$$

construct the area element

$$dz_1^* dz_2 = |dz_1||dz_2|e^{i(\theta_2-\theta_1)}. \tag{3.122}$$

The transformed quantities are

$$df_1 = \frac{df}{dz_1}dz_1, \qquad df_2 = \frac{df}{dz_2}dz_2. \tag{3.123}$$

Using the definition of an analytic function we have

$$\frac{df}{dz_1} = \frac{df}{dz_2} = \frac{df}{dz} \tag{3.124}$$

that depends only on z. Let

$$\frac{df}{dz} = \left|\frac{df}{dz}\right| e^{i\alpha}. \tag{3.125}$$

Thus show that

$$df_1^* df_2 = \left|\frac{df}{dz}\right|^2 |dz_1||dz_2|e^{i(\theta_2-\theta_1)}. \tag{3.126}$$

Thus conclude that the shape $df_1^* df_2$ and $dz_1^* dz_2$ preserve angles.

3.3 Analytic function

1. (**30 points.**) Analytic functions are significantly constrained, in that they have to satisfy the Cauchy-Riemann conditions. These conditions are necessary (but not sufficient) for a function of a complex variable to be analytic (differentiable). Check if the following functions satisfy the Cauchy-Riemann conditions. If $f(z)$ is analytic for all z, then report the derivative as a function of z. Otherwise, determine the points, or regions, in the z plane where the function is not analytic.

$$f(z) = z^3, \tag{3.127a}$$
$$f(z) = |z|^2, \tag{3.127b}$$
$$f(z) = e^{iz}, \tag{3.127c}$$
$$f(z) = \ln z. \tag{3.127d}$$

Use ComplexContourPlot in Mathematica to visualize these functions.

2. (**20 points.**) Check if the function

$$f(z) = zz^* \tag{3.128}$$

satisfies the Cauchy-Riemann conditions.

(a) Verify that all the points for $f(z)$ lies on the non-negative real line.

(b) Verify that as you approach the point $z = r \geq 0$ on the non-negative real line, along a circle of fixed radius r from the first quadrant, we have

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\theta \to 0} \frac{f(re^{i\theta}) - f(r)}{re^{i\theta} - r} = 0. \quad (3.129)$$

Then verify that as you approach the point $z = r$ along the real axis we have

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 + y^2 - (x^2 + y^2)}{\Delta x} = 2x. \quad (3.130)$$

(c) Thus, conclude that the derivative is not isotropic for any z.

(d) Use ComplexContourPlot in Mathematica to visualize these functions. Note that this is not an analytic function.

3. (**20 points.**) Check if the function

$$f(z) = e^{z+iz} \quad (3.131)$$

satisfies the Cauchy-Riemann conditions.

4. (**20 points.**) Check if the function

$$f(z) = e^z + e^{iz} \quad (3.132)$$

satisfies the Cauchy-Riemann conditions. If $f(z)$ is analytic for all z, then report the derivative as a function of z. Otherwise, determine the points, or regions, in the z plane where the function is not analytic.

5. (**20 points.**) Check if the function

$$f(z) = \frac{1}{z} \quad (3.133)$$

satisfies the Cauchy-Riemann conditions.

(a) Verify that the Cauchy-Riemann conditions for this case are not well defined at $z = 0$, but are fine for $z \neq 0$.

(b) Verify that

$$\frac{df}{dz} = -\frac{1}{z^2}, \qquad z \neq 0. \quad (3.134)$$

(c) Determine the limiting value of the derivative as you approach $z = 0$ along the positive real line, and, then, when you approach along the negative real line. Repeat the analysis along the imaginary line. Repeat the analysis along the line $x = y$. Are these limits identical?

(d) If these limits are not identical conclude that the derivative is not isotropic at $z = 0$. Then, the function is not analytic at $z = 0$.

6. (**20 points.**) Check if the function

$$f(z) = \ln \frac{(z-1)}{(z+1)} \quad (3.135)$$

satisfies the Cauchy-Riemann conditions. Investigate the geometric properties of this function using ComplexContourPlot in Mathematica.

7. (**20 points.**) Plot

$$f(x) = e^{-\frac{1}{x}}, \quad (3.136)$$

given x is positive and real. Also, imagine the plot for $f(iy) = e^{-\frac{1}{iy}}$. Let z be complex. Check if the function

$$f(z) = e^{-\frac{1}{z}} \quad (3.137)$$

satisfies the Cauchy-Riemann conditions.

(a) Verify that the Cauchy-Riemann conditions for this case are not well defined at $z = 0$, but are fine for $z \neq 0$.

(b) Verify that

$$\frac{df}{dz} = \frac{e^{-\frac{1}{z}}}{z^2}, \qquad z \neq 0. \tag{3.138}$$

(c) Determine the limiting value of the derivative as you approach $z = 0$ along the positive real line, and, then, when you approach along the negative real line. Repeat the analysis along the imaginary line. Repeat the analysis along the line $x = y$. Are these limits identical?

(d) If these limits are not identical conclude that the derivative is not isotropic at $z = 0$. Then, the function is not analytic at $z = 0$.

8. (**20 points.**) Plot

$$f(x) = e^{-\frac{1}{1-x^2}}, \tag{3.139}$$

given x is real. Let z be complex. Check if the function

$$f(z) = e^{-\frac{1}{1-z^2}} \tag{3.140}$$

satisfies the Cauchy-Riemann conditions. In particular, investigate when $z = 0$ and $z \neq 0$.
Hint: Inquire whether $1/(1 - z^2)$ is analytic.

9. (**20 points.**) (Under construction.
) Plot

$$f(x) = \frac{\sin x}{x} \tag{3.141}$$

given x is real. Let z be complex. Is the complex function

$$f(z) = \frac{\sin z}{z} \tag{3.142}$$

analytic at $z = 0$? Show that

$$\frac{df}{dz} = \frac{z \cos z - \sin z}{z^2}. \tag{3.143}$$

Query: Doesn't df/dz approach zero from all directions? It seems to. For example,

$$\lim_{x\to 0}\lim_{y\to 0} \frac{z \cos z - \sin z}{z^2} = \lim_{y\to 0}\lim_{x\to 0} \frac{z \cos z - \sin z}{z^2}. \tag{3.144}$$

However, literature seems to suggest that $\sin z/z$ is not analytic at $z = 0$. These discussions should be verified with the statement of Morera's theorem in the integral of $\sin z/z$ being π.

3.4 Conjugate functions: Electrostatics in two dimensions

1. (**20 points.**) Given

$$f = u + iv, \tag{3.145}$$

where

$$u(x, y) = x^2 - y^2 \tag{3.146}$$

and the function $v(x, y)$ is not known. If the function f is an analytic function, then v satisfies the Cauchy-Riemann conditions. Determine v, assuming f is analytic.

2. (**20 points.**) Show that the complex function

$$f = zz^* \tag{3.147}$$

is not an analytic function. Express the function f in the form $f = u + iv$ and find that $u = (x^2 + y^2)$ and $v = 0$. Show that the Cauchy-Riemann conditions are not satisfied in this case. Could we modify the function v, keeping u the same, such that the new function f is analytic? That is, investigate if there exists a function v that satisfies the Cauchy-Riemann conditions with $u = (x^2 + y^2)$? If yes, find a v and interpret? If no, what is the interpretation?

3. (**20 points.**) Let

$$f(z) = z^3, \tag{3.148}$$

so that

$$u(x, y) + iv(x, y) = r^3(\cos 3\theta + \sin 3\theta). \tag{3.149}$$

(a) Verify that this function satisfies the Cauchy-Riemann conditions.

(b) Show that u and v are harmonic functions. That is, they satisfy the Laplacian. Further, show that

$$(\boldsymbol{\nabla} u) \cdot (\boldsymbol{\nabla} v) = 0. \tag{3.150}$$

Thus, the curves represented by u and v are orthogonal at every point.

(c) Since u is a harmonic function it represents equipotential curves. Plot the equipotentials

$$r = \left[\frac{u}{\cos 3\theta}\right]^{\frac{1}{3}} \tag{3.151}$$

for $u = -10, -1, -0.1, 0, 0.1, 1, 10$. In Mathematica this can be achieved using the command

```
PolarPlot[{r[-10],...,r[10]},{th,0,2 Pi}],
```

where `r[u]` a function of u and th needs to be defined ahead.

(d) Determine the electric field associated to these equipotentials using

$$\mathbf{E} = -\boldsymbol{\nabla} u. \tag{3.152}$$

This is easily achieved using

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} \tag{3.153}$$

and similarly for derivatives with respect to y. Recall

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}. \tag{3.154}$$

Show that

$$\mathbf{E} = -\hat{\mathbf{i}}\, 3r^2 \cos 2\theta + \hat{\mathbf{j}}\, 3r^2 \sin 2\theta. \tag{3.155}$$

(e) The curves representing the field lines are obtained by requiring the tangent lines for these curves to have the same slope as the electric field, $\mathbf{E} = (\text{constant})\, d\mathbf{s}$, where $d\mathbf{s} = \hat{\mathbf{i}}dx + \hat{\mathbf{j}}dy + \hat{\mathbf{k}}dz$, such that

$$\frac{dx}{E_x} = \frac{dy}{E_y}. \tag{3.156}$$

Rewrite this equation as

$$E_y dx - E_x dy = 0. \tag{3.157}$$

Comparing this equation with

$$\frac{\partial s}{\partial x}\,dx + \frac{\partial s}{\partial y}\,dy = 0 \tag{3.158}$$

identify the equations satisfied by the curves $s(x,y)$, representing the field lines associated to the equipotentials u, as

$$\frac{\partial s}{\partial x} = 6xy, \qquad \frac{\partial s}{\partial y} = 3(x^2 - y^2). \tag{3.159}$$

Solve these equations to determine the equations for the field lines to be

$$s(x,y) = 3x^2y - y^3 = r^3 \sin 3\theta \tag{3.160}$$

up to a constant. The field lines s are indeed v. Plot the field lines

$$r = \left[\frac{v}{\sin 3\theta}\right]^{\frac{1}{3}} \tag{3.161}$$

for $v = -10, -1, -0.1, 0, 0.1, 1, 10$.

(f) Plot the equipotentials in red and field lines in blue in the same plot. Here is a simple code for it in Mathematica

```
n = 3;
f[u_] = (u/Cos[n t])^(1/n);
g[u_] = (u/Sin[n t])^(1/n);
PolarPlot[
 {f[-10], f[-1], f[-0.1], f[0], f[0.1], f[1], f[10],
  g[-10], g[-1], g[-0.1], g[0], g[0.1], g[1], g[10]},
 {t, -Pi, Pi},
 PlotStyle -> {Red, Red, Red, Red, Red, Red, Red,
              Blue, Blue, Blue, Blue, Blue, Blue, Blue},
 PlotRange -> {-4, 4}]
```

which generates the plots in Fig. 3.1.

4. (**20 points.**) Let

$$f(z) = -\lambda \ln z, \tag{3.162}$$

λ being positive real, so that

$$u(x,y) + iv(x,y) = -\lambda(\ln r + i\theta). \tag{3.163}$$

If u's are interpreted as equipotential surfaces, this represents the electrostatic configuration consisting of a line charge of strength λ along the line $z = 0$. Determine the electrostatic configuration corresponding to the complex function

$$f(z) = -\lambda \ln \frac{z - z_0}{z + z_0}. \tag{3.164}$$

(a) In particular, show that

$$f(z) = -\lambda \ln(z - z_0) + \lambda \ln(z + z_0). \tag{3.165}$$

Thus, interpret the electrostatic configuration to consist of a line dipole. That is, it consists of a line charge of strength λ along the line $z = z_0$, and another line charge of strength $-\lambda$ along the line $z = -z_0$.

(b) Evaluate $f(z)$ in the limit $2z_0 \to 0$, $\lambda \to \infty$, such that the product $2z_0\lambda = p$ is kept fixed. In particular, show that

$$f(z) \to \frac{p}{z}. \tag{3.166}$$

Interpret this configuration to be that of a line point-dipole of strength p (with direction given by the position of the complex number p in the complex plane) along the line passing through the origin.

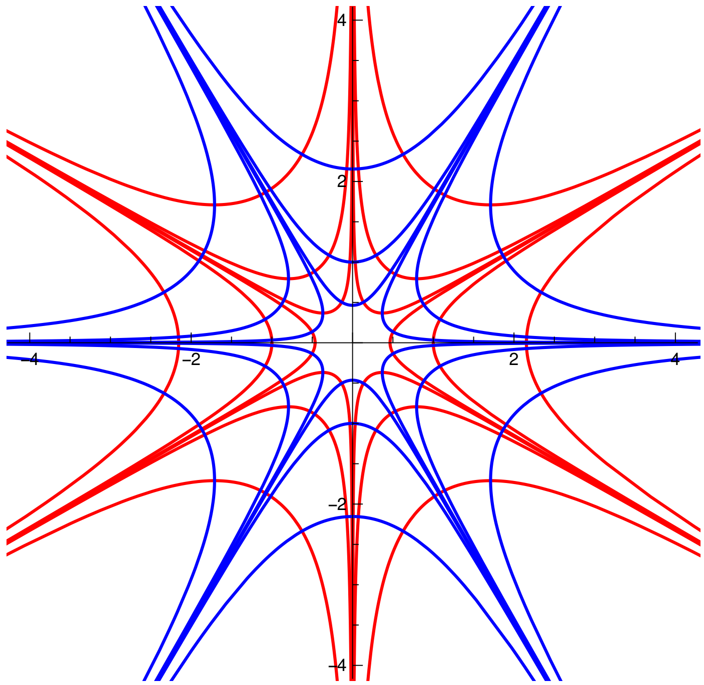

Figure 3.1: Equipotentials and field lines represented by the analytic function $f(z) = z^3$.

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3.5 Cauchy theorem

3.5.1 Cauchy integral formula

For a complex function that is analytic everywhere inside a contour c we have

$$\frac{1}{2\pi i}\oint_c dz \frac{f(z)}{(z-a)} = \begin{cases} f(a), & \text{if } a \text{ is inside the region enclosed by contour } c, \\ 0, & \text{otherwise.} \end{cases} \tag{3.167}$$

1. (**10 points.**) Evaluate the following contour integrals. In the following the contour c is a unit circle going counterclockwise with center at $z = a$.

$$I(a) = \frac{1}{2\pi i}\oint_c dz \frac{(z^5+1)}{(z-a)}, \tag{3.168a}$$

$$I(a) = \frac{1}{2\pi i}\oint_c dz \frac{e^{iz}}{(z-a)}. \tag{3.168b}$$

2. (**20 points.**) Evaluate the contour integral

$$I = \frac{1}{2\pi i}\oint_c dz \frac{e^{iz}}{(z^2-a^2)}, \tag{3.169}$$

where the contour c is a unit circle going counterclockwise with center at the origin. Inquire the cases when $|a| > 1$ and $|a| < 1$.

3.5.2 Cauchy differentiation formula

$$\frac{1}{2\pi i}\oint_c dz \frac{f(z)}{(z-a)^{n+1}} = \frac{1}{n!}\left\{\frac{d^n f(z)}{dz^n}\right\}_{z=a}, \qquad n = 0, 1, 2, \ldots. \tag{3.170}$$

Since f(z) is analytic there exists a power series expansion of $f(z)$ about $z = a$. The, use the Cauchy integral formula.

3.6 Laurent series

Laurent series for a function $f(z)$, about $z = z_0$, involves the expansion

$$f(z) = \ldots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots, \tag{3.171}$$

where the coefficients a_n are given using the contour integrals

$$a_n = \frac{1}{2\pi i}\oint_c dz \frac{f(z)}{(z-z_0)^{n+1}} \tag{3.172}$$

with the contour c encircling z_0.

1. (**20 points.**) The complex function

$$f(z) = \frac{1}{(z+2)(z-1)} \tag{3.173}$$

has the Laurent series

$$f(z) = \ldots + \frac{a_{-2}}{(z+2)^2} + \frac{a_{-1}}{(z+2)} + a_0 + a_1(z+2) + a_2(z+2)^2 + \ldots, \tag{3.174}$$

about $z = -2$, where

$$a_n = \frac{1}{2\pi i}\oint_c dz \frac{1}{(z+2)^{n+1}} \frac{1}{(z+2)(z-1)}. \tag{3.175}$$

Choose the contour c to be a circle centered at $z = -2$ with radius less than 3 so that it does not encircle $z = 1$. Show that

$$a_n = \begin{cases} 0, & \text{if } n = -2, -3, -4, \ldots, \\ \dfrac{1}{(n+1)!}\left\{\dfrac{d^{n+1} f(z)}{dz^{n+1}}\right\}_{z=-2}, & \text{if } n = -1, 0, 1, 2, \ldots. \end{cases} \tag{3.176}$$

In particular, show that

$$a_n = \begin{cases} 0, & \text{if } n = -2, -3, -4, \ldots, \\ -\dfrac{1}{3^{n+2}}, & \text{if } n = -1, 0, 1, 2, \ldots. \end{cases} \tag{3.177}$$

2. (**20 points.**) The complex function

$$f(z) = \frac{1}{(z+2)(z-1)} \tag{3.178}$$

has the Laurent series

$$f(z) = \ldots + \frac{a_{-2}}{(z+2)^2} + \frac{a_{-1}}{(z+2)} + a_0 + a_1(z+2) + a_2(z+2)^2 + \ldots, \tag{3.179}$$

about $z = -2$, where

$$a_n = \frac{1}{2\pi i}\oint_c dz \frac{f(z)}{(z+2)^{n+1}}. \tag{3.180}$$

Choose the contour c to be a circle centered at $z = -2$ with radius less than 3 so that it does not encircle $z = 1$. Find a_n. Then, discuss the case when the contour encircles $z = 1$.

3. (**20 points.**) Find the Laurent series for

$$f(z) = \frac{1}{(z+2)(z-1)} \tag{3.181}$$

about $z = 1$.

4. (**20 points.**) Show that the coefficients a_n in the Laurent series for the complex function

$$e^{\frac{x}{2}(z-\frac{1}{z})} \tag{3.182}$$

are Bessel functions of order n. It is called the generating function of Bessel functions.

3.7 Contour integrals with poles

1. (**20 points.**) Evaluate the contour integral

$$I = \frac{1}{2\pi i}\oint_c \frac{dz}{z}, \tag{3.183}$$

where the contour c is a unit circle going counterclockwise with center at the origin.

2. (**20 points.**) Evaluate the contour integral

$$I = \frac{1}{2\pi i}\oint_c \frac{dz}{z^2}, \tag{3.184}$$

where the contour c is a unit circle going counterclockwise with center at the origin.

3. (**Example.**) The Heaviside step function is defined as

$$\theta(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases} \tag{3.185}$$

What is the Fourier transform of the Heaviside step function? Recall the Fourier transform and the corresponding inverse,

$$\theta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\tilde{\theta}(\omega), \tag{3.186a}$$

$$\tilde{\theta}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\theta(t). \tag{3.186b}$$

(a) Using the definition in Eq. (5.32) in Eq. (5.33b) show that

$$\tilde{\theta}(\omega) = \int_0^{\infty} dt\, e^{i\omega t} = \lim_{\delta\to 0+}\int_0^{\infty} dt\, e^{i\omega t}e^{-\delta t} = \lim_{\delta\to 0+} -\frac{1}{i}\frac{1}{\omega + i\delta}. \tag{3.187}$$

(b) Verify that

$$\theta(t) = \lim_{\delta\to 0+} -\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\omega\, \frac{e^{-i\omega t}}{\omega + i\delta} \tag{3.188}$$

is indeed an integral representation of Heaviside step function.

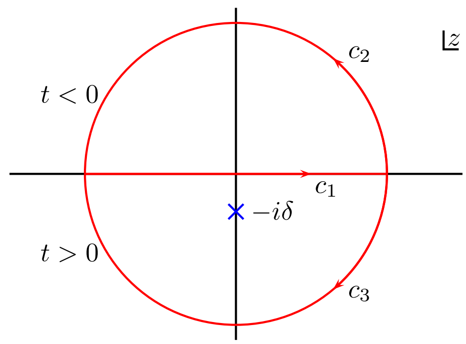

Figure 3.2: Contour $c = c_1 + c_2$ for $t < 0$ and the contour $c = c_1 + c_3$ for $t > 0$ used to evaluate the Heaviside step function.

i. Let $t < 0$. Let us consider

$$G_1(t) = \lim_{\delta \to 0+} -\frac{1}{2\pi i} \oint_{c_1 + c_2} dz \, \frac{e^{-izt}}{z + i\delta}. \tag{3.189}$$

See Figure 3.2. Using Cauchy's theorem show that

$$G_1(t) = 0. \tag{3.190}$$

For the part of contour constituting c_1 substitute $z = x$ and show that

$$\lim_{\delta \to 0+} -\frac{1}{2\pi i} \int_{c_1} dz \, \frac{e^{-izt}}{z + i\delta} = \theta(t). \tag{3.191}$$

For the part of contour constituting c_2 substitute $z = Re^{i\theta}$ and show that

$$\lim_{R \to \infty} \lim_{\delta \to 0+} -\frac{1}{2\pi i} \int_{c_2} dz \, \frac{e^{-izt}}{z + i\delta} = 0. \tag{3.192}$$

Thus, together, conclude that

$$\theta(t) = 0, \qquad t < 0. \tag{3.193}$$

ii. Let $t > 0$. Let us consider

$$G_2(t) = \lim_{\delta \to 0+} -\frac{1}{2\pi i} \oint_{c_1 + c_3} dz \, \frac{e^{-izt}}{z + i\delta}, \tag{3.194}$$

where note that the contour is going clockwise, opposite of the convention used in Cauchy's theorem. See Figure 3.2. Using Cauchy's theorem show that

$$G_2(t) = 1. \tag{3.195}$$

For the part of contour constituting c_1 substitute $z = x$ and show that

$$\lim_{\delta \to 0+} -\frac{1}{2\pi i} \int_{c_1} dz \, \frac{e^{-izt}}{z + i\delta} = \theta(t). \tag{3.196}$$

For the part of contour constituting c_3 substitute $z = Re^{i\theta}$ and show that

$$\lim_{R \to \infty} \lim_{\delta \to 0+} -\frac{1}{2\pi i} \int_{c_3} dz \, \frac{e^{-izt}}{z + i\delta} = 0. \tag{3.197}$$

Thus, together, conclude that

$$\theta(t) = 1, \qquad t > 0. \tag{3.198}$$

(c) Observe that $G_1(t)$ and $G_2(t)$ are equal on the real axis, but are not equal in general. However, the analyticity of these functions allows us to deduce an integral on the real line in terms of the value of the functions in different regions of the complex plane.

4. (**20 points.**) See Figure 3.2. Let $\delta > 0$.

   (a) For $t < 0$ evaluate

$$G_1(t) = -\frac{1}{2\pi i}\oint_{c_1+c_2} dz\,\frac{e^{-izt}}{z+i\delta}. \tag{3.199}$$

   (b) For $t > 0$ evaluate

$$G_2(t) = -\frac{1}{2\pi i}\oint_{c_1+c_3} dz\,\frac{e^{-izt}}{z+i\delta}, \tag{3.200}$$

   where note that the contour is going clockwise, opposite of the convention used in Cauchy's theorem.

5. (**20 points.**) Evaluate the integral

$$I(a) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dx}{(x+ia)} \tag{3.201}$$

   for $a > 0$. Do this by extending to the complex plane and evaluating along a contour in the upper half complex plane. Repeat the exercise along a contour in the lower half complex plane.
   Solution: $-1/2$.

6. (**20 points.**) Evaluate the integral

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+1} \tag{3.202}$$

   using Cauchy's theorem, after choosing a suitable contour. Verify your result by evaluating the integral using the elementary substitution method, $x = \tan\theta$.
   Solution: π.

7. (**20 points.**) Evaluate the integral

$$\int_{-\infty}^{\infty}\frac{dx\, e^{iax}}{x^2+1} \tag{3.203}$$

   using Cauchy's theorem, after choosing a suitable contour. Here a is real.
   Solution: $\pi e^{-|a|}$.

8. (**20 points.**) Evaluate the integral

$$\int_{-\infty}^{\infty} dx\frac{\sin x}{x} \tag{3.204}$$

   using Cauchy's theorem, after choosing a suitable contour. The complex function $\sin z/z$ has a pole at $z = 0$ and it lies on the contour. To avoid this, consider the integral

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dx\frac{\sin x}{x+i\epsilon}. \tag{3.205}$$

   To verify the result consider the integral

$$I(a) = \int_{-\infty}^{\infty} dx\frac{\sin x}{x}e^{-ax}. \tag{3.206}$$

   Evaluate $-dI/da$. Then, integrate with the conditions $I(\infty) = 0$ to evaluate $I(0)$.
   Solution: π.

9. (**20 points.**) Consider the integral

$$I(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(1 + a\cos\theta)}, \tag{3.207}$$

where a is complex. Substitute $z = e^{i\theta}$, such that

$$2\cos\theta = z + \frac{1}{z}, \tag{3.208}$$

and express the integral as a contour integral,

$$I(a) = \frac{1}{2\pi i}\frac{2}{a}\oint_c \frac{dz}{\left(z^2 + \frac{2}{a}z + 1\right)}, \tag{3.209}$$

where the contour c is along the unit circle going counterclockwise. Show that

$$z^2 + \frac{2}{a}z + 1 = (z - r_+)(z - r_-), \tag{3.210}$$

where

$$r_\pm = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - 1}. \tag{3.211}$$

Using residue theorem evaluate $I(a)$ for $|\mathrm{Re}(a)| < 1$ and $\mathrm{Im}(a) = 0$.

10. (**20 points.**) Consider the contour integral

$$I(a) = \frac{1}{2\pi i}\frac{2}{a}\oint_c \frac{dz}{\left(z^2 + \frac{2}{a}z + 1\right)}, \tag{3.212}$$

where the contour c is along the unit circle going counterclockwise. Show that

$$z^2 + \frac{2}{a}z + 1 = (z - r_+)(z - r_-), \tag{3.213}$$

where

$$r_\pm = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - 1}. \tag{3.214}$$

Using residue theorem evaluate $I(\frac{1}{3})$.

11. (**20 points.**) Consider the contour integral

$$I(v, w) = \frac{1}{2\pi i}\frac{1}{2}\oint_c \frac{dz}{z}\frac{z^2 + 2\frac{w}{v}z + 1}{\left(z + \frac{v}{w}\right)\left(z + \frac{w}{v}\right)}, \tag{3.215}$$

where the contour c is along the unit circle going counterclockwise. Evaluate $I(1, 2)$ and $I(2, 1)$. In general, what happens when $v < w$ and $v > w$?

12. (**90 points.**) Consider the integral

$$I(v, w) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{w^2 + vw\cos\theta}{v^2 + w^2 + 2vw\cos\theta}, \tag{3.216}$$

where v and w are complex.

(a) Substitute $z = e^{i\theta}$, such that

$$2\cos\theta = z + \frac{1}{z}, \tag{3.217}$$

and express the integral as a contour integral,

$$I(v,w) = \frac{1}{2\pi i}\frac{1}{2}\oint_c \frac{dz}{z}\frac{z^2 + 2\frac{w}{v}z + 1}{\left(z+\frac{v}{w}\right)\left(z+\frac{w}{v}\right)}, \tag{3.218}$$

where the contour c is along the unit circle going counterclockwise. Locate the three poles, $z = 0$, $z = -v/w$, and $z = -w/v$.

(b) Evaluate the residues and show that

$$I(v,w) = \begin{cases} 1, & \text{if } |v| < |w|, \\ 0, & \text{if } |w| < |v|. \end{cases} \tag{3.219}$$

Observe that for $v = w$, (which is more restrictive than $|v| = |w|$,) we have

$$I(v,w) = \frac{1}{2}. \tag{3.220}$$

(c) Let us seek the partial fraction decomposition

$$\frac{z^2 + 2\frac{w}{v}z + 1}{z\left(z+\frac{v}{w}\right)\left(z+\frac{w}{v}\right)} = \frac{a}{z} + \frac{b}{\left(z+\frac{v}{w}\right)} + \frac{c}{\left(z+\frac{w}{v}\right)}. \tag{3.221}$$

Show that $a = 1$, $b = 1$, and $c = -1$. Thus, express the integral in the form

$$I(v,w) = \frac{1}{2}\left[\frac{1}{2\pi i}\oint_c \frac{dz}{z} + \frac{1}{2\pi i}\oint_c \frac{dz}{\left(z+\frac{v}{w}\right)} - \frac{1}{2\pi i}\oint_c \frac{dz}{\left(z+\frac{w}{v}\right)}\right]. \tag{3.222}$$

(d) Show that

$$\frac{1}{2\pi i}\oint_c \frac{dz}{z} = 1. \tag{3.223}$$

Evaluate the integrals

$$\frac{1}{2\pi i}\oint_c \frac{dz}{\left(z+\frac{v}{w}\right)} = \theta\left(1 - \frac{|v|}{|w|}\right), \tag{3.224a}$$

$$\frac{1}{2\pi i}\oint_c \frac{dz}{\left(z+\frac{w}{v}\right)} = \theta\left(1 - \frac{|w|}{|v|}\right), \tag{3.224b}$$

where $\theta(x)$ is the Heaviside step function. Thus, derive the relation

$$I(v,w) = \frac{1}{2}\left[1 + \theta\left(1 - \frac{|v|}{|w|}\right) - \theta\left(1 - \frac{|w|}{|v|}\right)\right] \tag{3.225}$$

and verify Eq. (3.219).

(e) What electrostatic configuration in two dimensions represents the complex function in Eq. (3.221).

13. (**20 points.**) Consider the integral

$$I(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(1 + a\cos\theta)}, \tag{3.226}$$

where a is complex.

(a) Substitute $z = e^{i\theta}$, such that

$$2\cos\theta = z + \frac{1}{z}, \tag{3.227}$$

and express the integral as a contour integral,

$$I(a) = \frac{1}{2\pi i}\frac{2}{a}\oint_c \frac{dz}{\left(z^2 + \frac{2}{a}z + 1\right)}, \tag{3.228}$$

where the contour c is along the unit circle going counterclockwise. Show that

$$z^2 + \frac{2}{a}z + 1 = (z - r_+)(z - r_-), \tag{3.229}$$

where

$$r_\pm = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - 1}. \tag{3.230}$$

(b) Show that for real a

$$|r_\pm| \begin{cases} < 1, & \text{if } |\mathrm{Re}(a)| < 1 \text{ and } \mathrm{Im}(a) = 0, \\ = 1, & \text{if } |\mathrm{Re}(a)| \geq 1 \text{ and } \mathrm{Im}(a) = 0. \end{cases} \tag{3.231}$$

Let $a = |a|e^{i\alpha}$. Investigate the position of $r_\pm$ on the complex plane. Verify (by plotting absolute value of $r_\pm$) that

$$|r_+| \begin{cases} < 1, & \text{if } \mathrm{Re}(a) > 0 \text{ and } \mathrm{Im}(a) \neq 0, \\ > 1, & \text{if } \mathrm{Re}(a) \leq 0 \text{ and } \mathrm{Im}(a) \neq 0, \end{cases} \tag{3.232}$$

and

$$|r_-| \begin{cases} > 1, & \text{if } \mathrm{Re}(a) > 0 \text{ and } \mathrm{Im}(a) \neq 0, \\ < 1, & \text{if } \mathrm{Re}(a) \leq 0 \text{ and } \mathrm{Im}(a) \neq 0. \end{cases} \tag{3.233}$$

Thus, locate the two poles, $z = r_\pm$, for complex values of a.

(c) Evaluate the residues and show that

$$I(a) = \begin{cases} \dfrac{1}{\sqrt{1-a^2}}, & \text{if } |\mathrm{Re}(a)| < 1 \text{ and } \mathrm{Im}(a) = 0, \\ \dfrac{1}{\sqrt{1-a^2}}, & \text{if } \mathrm{Re}(a) > 0 \text{ and } \mathrm{Im}(a) \neq 0, \\ -\dfrac{1}{\sqrt{1-a^2}}, & \text{if } \mathrm{Re}(a) < 0 \text{ and } \mathrm{Im}(a) \neq 0, \\ \text{divergent}, & \text{if } |\mathrm{Re}(a)| \geq 1 \text{ and } \mathrm{Im}(a) = 0. \end{cases} \tag{3.234}$$

14. (**20 points.**) Consider the integral

$$I(a) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{1}{1 - 2a\cos\theta + a^2}, \tag{3.235}$$

where a is complex.

(a) Substitute $z = e^{i\theta}$, such that

$$2\cos\theta = z + \frac{1}{z}, \tag{3.236}$$

and express the integral as a contour integral along the unit circle going counterclockwise. Locate the poles.

(b) Evaluate the residues and show that

$$I(a) = \begin{cases} \frac{1}{1-a^2}, & \text{if } |a| < 1, \\ \frac{1}{a^2-1}, & \text{if } |a| > 1. \end{cases} \tag{3.237}$$

(c) Plot $I(a)$ for real values of a. Plot real and imaginary part of $I(a)$ for complex a. Argue that $I(1)$ is divergent.

15. (**20 points.**) Evaluate the contour integral

$$\oint_c dz \frac{(z^5 + z^3 + 1)}{(z^2 - 5z + 6)}, \tag{3.238}$$

where the contour c is along the unit circle going counterclockwise.

3.8 Contour integrals with branch points

1. (**20 points.**) Show that

$$\oint_{c1} dz \ln z = 2\pi i R, \tag{3.239a}$$

$$\oint_{c2} dz \ln z = 0, \tag{3.239b}$$

where the contours $c1$ and $c2$ are shown in Figure 3.3, and R is the radius of the circle forming the contour. Is the function $\ln z$ analytic at $z = 0$? Is the function $\ln z$ analytic at $z \neq 0$? Show that if the contour c winds around the origin more than once the integral evaluates to

$$\oint_c dz \ln z = 2\pi i R n, \tag{3.240}$$

where n is the number of times the contour winds around the origin.
Hint: Show that

$$\oint_{c1} dz \ln z = -R \int_0^{2\pi} \theta d\theta e^{i\theta}. \tag{3.241}$$

2. (**20 points.**) Discuss the discontinuities (branch cut) in the complex function

$$f(z) = \ln z \tag{3.242}$$

on the complex plane z. In particular, qualitatively discuss if the contour integrals

$$\oint_{c1} dz \ln z \quad \text{and} \quad \oint_{c2} dz \ln z \tag{3.243}$$

evaluate to zero using Cauchy's theorem, where the contours $c1$ and $c2$ are shown in Figure 3.3. Show that

$$\oint_{c1} dz \ln z = 2\pi i r, \tag{3.244a}$$

$$\oint_{c2} dz \ln z = 0. \tag{3.244b}$$

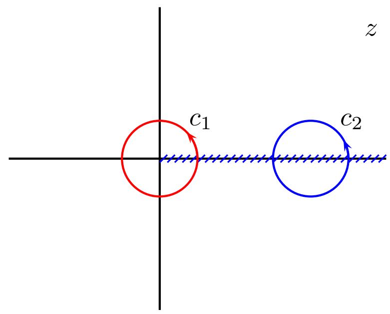

Figure 3.3: Contour c_1 encircles the origin while contour c_2 does not encircle the origin.

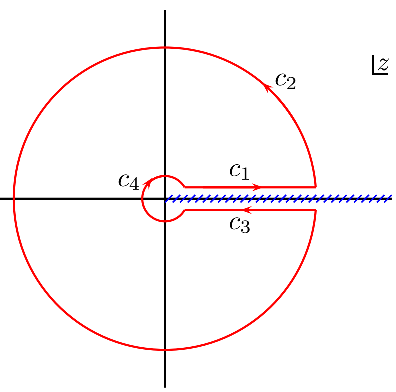

Figure 3.4: Contour $c = c_1 + c_2 + c_3 + c_4$. The radii of the contours c_2 and c_4 are R and ϵ, respectively, and contours c_1 and c_3 are δ away from the real line. We assume limits $\epsilon \to 0$, $R \to \infty$, and $\delta \to 0$.

3. (**Example.**) Consider the integral

$$I(\theta) = \frac{1}{\pi} \int_0^\infty \frac{x^{\frac{1}{2}}\, dx}{1 + 2x\cos\theta + x^2}, \tag{3.245}$$

where $0 \le \theta < 2\pi$ is real. To evaluate $I(\theta)$ let us consider the following integral on the complex plane

$$G(\theta) = \frac{1}{\pi} \oint_c \frac{z^{\frac{1}{2}}\, dz}{1 + 2z\cos\theta + z^2}, \tag{3.246}$$

where the contour c is described in Figure 3.4.

(a) Show that

$$1 + 2z\cos\theta + z^2 = (z + e^{i\theta})(z + e^{-i\theta}) \tag{3.247}$$

and identify the poles. Show that the integrand has a branch point at $z = 0$. Choose the branch cut to be the positive real line. Using Cauchy's theorem show that

$$G(\theta) = 2\frac{\sin\frac{\theta}{2}}{\sin\theta}. \tag{3.248}$$

(b) Next, let us evaluate $G(\theta)$ by evaluating the integrals on the contour explicitly.

i. For the part of contour constituting c_1 substitute $z = xe^{i\delta} \sim x + i\delta'$ and show that

$$\lim_{\delta\to 0} \lim_{\epsilon\to 0} \lim_{R\to\infty} \frac{1}{\pi} \oint_{c_1} \frac{z^{\frac{1}{2}}\, dz}{(z + e^{i\theta})(z + e^{-i\theta})} = I(\theta). \tag{3.249}$$

ii. For the part of contour constituting c_3 substitute $z = xe^{i(2\pi-\delta)} \sim x - i\delta'$ and show that

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\lim_{R\to\infty}\frac{1}{\pi}\oint_{c_3}\frac{z^{\frac{1}{2}}\,dz}{(z+e^{i\theta})(z+e^{-i\theta})} = -e^{i\frac{2\pi}{2}}I(\theta) = I(\theta). \tag{3.250}$$

iii. For the part of contour constituting c_2 substitute $z = Re^{i\theta}$ and show that

$$\lim_{R\to\infty}\frac{1}{\pi}\oint_{c_2}\frac{z^{\frac{1}{2}}\,dz}{(z+e^{i\theta})(z+e^{-i\theta})} = 0. \tag{3.251}$$

iv. For the part of contour constituting c_4 substitute $z = \epsilon e^{i\theta}$ and show that

$$\lim_{\epsilon\to 0}\frac{1}{\pi}\oint_{c_4}\frac{z^{\frac{1}{2}}\,dz}{(z+e^{i\theta})(z+e^{-i\theta})} = 0. \tag{3.252}$$

(c) Together, conclude that

$$2\frac{\sin\frac{\theta}{2}}{\sin\theta} = I(\theta) + 0 + I(\theta) + 0. \tag{3.253}$$

Thus, evaluate $I(\theta)$.

4. (**20 points.**) Evaluate the integral

$$I(\theta) = \frac{1}{\pi}\int_0^\infty \frac{x^{\frac{1}{3}}\,dx}{1+2x\cos\theta + x^2}, \tag{3.254}$$

where $0 \leq \theta < 2\pi$. Show that

$$1 + 2z\cos\theta + z^2 = (z+e^{i\theta})(z+e^{-i\theta}). \tag{3.255}$$

3.9 Analytic continuation

1. (**20 points.**) The following lecture recording from Fall 2020 available at

https://youtu.be/9Ac-en8ImDw

motivates the idea of analytic continuation. Let us consider the function

$$\mu(s) = \frac{1}{s}, \qquad s \neq 0. \tag{3.256}$$

(a) An integral representation of the function is

$$\mu(s) = \int_0^1 dt\, t^{s-1}, \qquad \text{Re}(s) > 0. \tag{3.257}$$

Evaluate the integral and show that the integral is indeed equal to $1/s$ for $\text{Re}(s) > 0$. However, the above integral representation breaks down for $\text{Re}(s) \leq 0$. Show that

$$\mu(0) = \int_0^1 \frac{dt}{t} = \lim_{\delta\to 0}\int_\delta^1 \frac{dt}{t} = -\lim_{\delta\to 0}\ln\delta \tag{3.258}$$

is logarithmically divergent. Similarly, show that

$$\mu(-1) = \int_0^1 \frac{dt}{t^2} = \lim_{\delta\to 0}\int_\delta^1 \frac{dt}{t^2} = \lim_{\delta\to 0}\left[1 - \frac{1}{\delta}\right]\frac{1}{(-1)} \tag{3.259}$$

is divergent. Check out $\mu(-2)$.

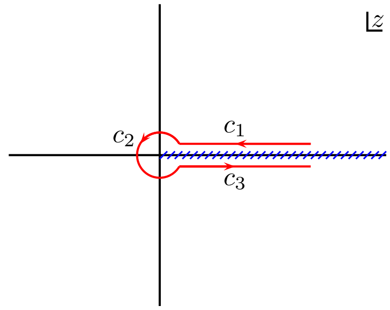

Figure 3.5: Contour $c = c_1 + c_2 + c_3$. The radius of the contour c_2 is ϵ and contours c_1 and c_3 are δ away from the real line. We assume limits $\epsilon \to 0$ and $\delta \to 0$.

(b) Another representation of the function valid on the complete complex plane of s is

$$\mu(s) = \frac{1}{(e^{i2\pi s} - 1)} \int_c dz\, z^{s-1}, \quad s \neq 0, \tag{3.260}$$

where the integral is evaluated on the contour $c = c_1 + c_2 + c_3$ described in Figure 3.5. Since the integral representation in Eq. (3.260) does not have the restriction $\text{Re}(s) > 0$, and because its values are identical to the integral representation in Eq. (3.257) for $\text{Re}(s) > 0$, it is the analytic continuation of the integral representation in Eq. (3.257).

i. For contour c_1 substitute $z = x\, e^{i\delta} \sim x + ix\delta$ and show that

$$\int_{c_1} dz\, z^{s-1} = \frac{1}{s}\Big(\epsilon^s - 1\Big). \tag{3.261}$$

ii. For contour c_3 substitute $z = \epsilon\, e^{i\theta}$ and show that

$$\int_{c_2} dz\, z^{s-1} = \frac{1}{s}\Big(e^{i2\pi s} - 1\Big)\epsilon^s. \tag{3.262}$$

iii. For contour c_3 substitute $z = x\, e^{i(2\pi - \delta)}$ and show that

$$\int_{c_3} dz\, z^{s-1} = \frac{1}{s}\Big(1 - \epsilon^s\Big)e^{i2\pi s}. \tag{3.263}$$

Together, we have

$$\mu(s) = \frac{1}{(e^{i2\pi s} - 1)} \frac{1}{s} \left[\Big(\epsilon^s - 1\Big) + \Big(e^{i2\pi s} - 1\Big)\epsilon^s + \Big(1 - \epsilon^s\Big)e^{i2\pi s}\right] = \frac{1}{s}. \tag{3.264}$$

Observe that the apparent divergence when the factor $(e^{i2\pi s} - 1)$ equals 0 for integer s is nonexistent.

3.10 List of topics

1. In electrostatics the static condition can be released by letting the curl to be non-zero, as a perturbation. In this spirit, can we construct a 'weakly' analytic function?

2. Mobius transformation, inversion, electrostatics, harmonic functions.

3. Continued fractions and power series. Can continued fraction of a number be derived using the idea of complex numbers? Or, using electrostatics.

4. Has complex functions on the surface of a sphere been studied?

Chapter 4

Matrix algebra and glimpse of quantum mechanics

4.1 Refer notes on quantum mechanics

Chapter 5

Function spaces

5.1 Vector space

1. A vector $\mathbf{A}$ in three dimensions can be expressed in the form

$$\mathbf{A} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3. \tag{5.1}$$

Here $\hat{\mathbf{e}}_i$ are called the basis vectors and a_i are components of the vector along the basis vectors.

(a) Orthogonality relation: Let us assume that the basis vectors are orthogonal to each other. This is stated compactly as

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}, \qquad i, j = 1, 2, 3, \tag{5.2}$$

where δ_{ij} is the Kronecker delta symbol.

(b) Vector components: Taking the dot product with $\hat{\mathbf{e}}_1$ in each term in Eq. (5.1) we obtain

$$\mathbf{A} \cdot \hat{\mathbf{e}}_1 = a_1(\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1) + a_2(\hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_1) + a_3(\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_1). \tag{5.3}$$

Using the orthogonality relations between the basis vectors we immediately have

$$\mathbf{A} \cdot \hat{\mathbf{e}}_1 = a_1. \tag{5.4}$$

Similar relations can be derived for other components, and they can be together expressed in the form

$$\mathbf{A} \cdot \hat{\mathbf{e}}_i = a_i, \qquad i = 1, 2, 3. \tag{5.5}$$

(c) Completeness relation: Substituting the expressions for the vector components back in Eq. (5.1) we have

$$\mathbf{A} = (\mathbf{A} \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1 + (\mathbf{A} \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2 + (\mathbf{A} \cdot \hat{\mathbf{e}}_3)\hat{\mathbf{e}}_3 \tag{5.6a}$$

$$= \mathbf{A} \cdot \Big[\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3\Big], \tag{5.6b}$$

where the second equality is obtained by recognizing the common factor. Thus, the vector multiplied with the quantity inside square brackets returns back the vector. Since the multiplication involves a scalar dot product, the quantity in square brackets can not be a vector because then it will return a scalar. We identify it to be the unit dyadic. Thus,

$$\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3 = \mathbf{1}, \tag{5.7}$$

which is the completeness relation for the basis vectors.

5.2 Discrete Fourier series

1. Wiki article on convergence of Fourier series
2. Dirichlet kernel
3. Gibbs phenomenon

1. The Fourier space is spanned by the Fourier eigenfunctions

$$e^{im\phi}, \qquad m = 0, \pm 1, \pm 2, \ldots, \qquad 0 \leq \phi < 2\pi. \tag{5.8}$$

An arbitrary function $f(\phi)$ has the Fourier series representation

$$f(\phi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} a_m e^{im\phi}, \tag{5.9}$$

where $e^{im\phi}$ are the Fourier eigenfunctions and a_m are the respective Fourier components.

(a) Orthogonality relation: The Fourier eigenfunctions satisfy the orthogonality relation

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \, e^{-in\phi} e^{im\phi} = \delta_{mn}. \tag{5.10}$$

(b) Fourier components: Using the orthogonality relations we can find the Fourier components to be

$$a_m = \int_0^{2\pi} d\phi \, e^{-im\phi} f(\phi). \tag{5.11}$$

(c) Completeness relation: The Fourier eigenfunctions satisfy the completeness relation

$$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\phi} e^{-im\phi'} = \delta(\phi - \phi'). \tag{5.12}$$

(d) Differential equation: The Fourier eigenfunctions satisfy the differential equation

$$-\left[\frac{d^2}{d\phi^2} - m^2 \right] e^{im\phi} = 0. \tag{5.13}$$

(e) Green's function: The associated Green's function satisfies the equation

$$-\left[\frac{d^2}{d\phi^2} - m^2 \right] g(\phi, \phi') = \delta(\phi - \phi'). \tag{5.14}$$

Verify by substitution that

$$g(\phi, \phi') = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \frac{e^{in\phi} e^{-in\phi'}}{n^2 - m^2} \tag{5.15}$$

satisfies the Green function equation.

5.2.1 Problems

1. (**20 points.**) Determine all the Fourier components a_m for the following functions: $\cos\phi$, $\sin\phi$, $\cos^2\phi$, $\sin^2\phi$, $\cos^3\phi$, $\sin^3\phi$.

2. (**20 points.**) Determine the particular function $f(\phi)$ that has the Fourier components

$$a_m = 1 \tag{5.16}$$

for all m. That is, all the Fourier coefficients are contributing equally in the series.

3. (**20 points.**) To determine the Fourier components of $\tan\phi$ start from

$$\tan\phi = \frac{1}{i}\frac{e^{i\phi} - e^{-i\phi}}{e^{i\phi} + e^{-i\phi}} \tag{5.17}$$

and show that

$$\tan\phi = \frac{1}{i} + \sum_{m=1}^{\infty} e^{-2im\phi}\frac{2(-1)^m}{i}. \tag{5.18}$$

Thus, read out all the Fourier components. Similarly, find the Fourier components of $\cot\phi$.

4. (**20 points.**) Fourier series (or transformation) is defined as $(0 \leq \phi < 2\pi)$

$$f(\phi) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{im\phi} a_m, \tag{5.19}$$

where the coefficients a_m are determined using

$$a_m = \int_0^{2\pi} d\phi\, e^{-im\phi} f(\phi). \tag{5.20}$$

Determine all the Fourier components a_m for the function $\cos^3\phi$.

5.3 Continuous Fourier integral

1. The (continuous) Fourier space is spanned by the Fourier eigenfunctions

$$e^{ikx}, \qquad -\infty < k < \infty, \qquad -\infty < x < \infty. \tag{5.21}$$

An arbitrary function $f(x)$ has the Fourier series representation

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}\tilde{f}(k), \tag{5.22}$$

where e^{ikx} are the Fourier eigenfunctions and $\tilde{f}(k)$ are the respective Fourier components.

(a) Orthogonality relation: The Fourier eigenfunctions satisfy the orthogonality relation

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-ik'x} e^{ikx} = \delta(k - k'). \tag{5.23}$$

(b) Fourier components: Using the orthogonality relations we can find the Fourier components to be

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x). \tag{5.24}$$

(c) Completeness relation: The Fourier eigenfunctions satisfy the completeness relation

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} e^{-ikx'} = \delta(x - x'). \tag{5.25}$$

(d) Differential equation: The Fourier eigenfunctions satisfy the differential equation

$$-\left[\frac{d^2}{dx^2} - k^2\right] e^{ikx} = 0. \tag{5.26}$$

5.3.1 Problems

1. (**20 points.**) Find the Fourier transform of a Gaussian function

$$f(x) = e^{-ax^2}. \tag{5.27}$$

That is, evaluate the integral

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} e^{-ax^2}. \tag{5.28}$$

2. (**20 points.**) Find the Fourier transform of the function

$$f(x) = e^{-a|x|}. \tag{5.29}$$

That is, evaluate the integral

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} e^{-a|x|}. \tag{5.30}$$

Solution:

$$\tilde{f}(k) = \frac{2a}{a^2 + k^2}. \tag{5.31}$$

3. (**20 points.**) The Heaviside step function is defined as

$$\theta(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases} \tag{5.32}$$

The Fourier transform and the corresponding inverse ae,

$$\theta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{\theta}(\omega), \tag{5.33a}$$

$$\tilde{\theta}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \theta(t). \tag{5.33b}$$

(a) Using the definition in Eq. (5.32) in Eq. (5.33b) show that

$$\tilde{\theta}(\omega) = \int_0^{\infty} dt\, e^{i\omega t} = \lim_{\delta \to 0+} \int_0^{\infty} dt\, e^{i\omega t} e^{-\delta t} = \lim_{\delta \to 0+} -\frac{1}{i} \frac{1}{\omega + i\delta}. \tag{5.34}$$

(b) Verify that

$$\theta(t) = \lim_{\delta \to 0+} -\frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega\, \frac{e^{-i\omega t}}{\omega + i\delta} \tag{5.35}$$

is indeed an integral representation of Heaviside step function.

4. (**20 points.**) Fourier series (or transformation) is defined as $(-\infty < x < \infty)$

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} a(k), \tag{5.36}$$

where the coefficients $a(k)$ are determined using

$$a(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x). \tag{5.37}$$

(a) Show that

$$\frac{d^n f(x)}{dx^n} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} (ik)^n e^{ikx} a(k). \tag{5.38}$$

(b) Show that the differential equation

$$-\left(\frac{d^2}{dx^2} - \omega^2\right) f(x) = \delta(x) \tag{5.39}$$

in the Fourier space is the algebraic equation

$$(k^2 + \omega^2) a(k) = 1. \tag{5.40}$$

Thus, the solution to the differential equation is the Fourier transform of

$$a(k) = \frac{1}{\omega^2 + k^2}. \tag{5.41}$$

Show that

$$f(x) = \frac{e^{-\omega|x|}}{2\omega}. \tag{5.42}$$

5. (**20 points.**) Consider the inhomogeneous linear differential equation

$$\left(a\frac{d^2}{dx^2} + b\frac{d}{dx} + c\right) f(x) = \delta(x). \tag{5.43}$$

Use the Fourier transformation and the associated inverse Fourier transformation

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k), \tag{5.44a}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x), \tag{5.44b}$$

to show that the corresponding equation satisfied by $\tilde{f}(k)$ is algebraic. Find $\tilde{f}(k)$.

5.4 Half-range Fourier series

1. Dirac comb, periodic Green's function, completeness relation for half-range in Schwinger's EM Section 17.7.

1. The half-range Fourier space is spanned by the Fourier eigenfunctions

$$\sin m\phi, \qquad m = 1, 2, 3, \ldots, \qquad 0 \le \phi \le \pi. \tag{5.45}$$

An arbitrary function $f(\phi)$, for ϕ limited to half the range, has the half-range Fourier series representation

$$f(\phi) = \sum_{m=1}^{\infty} a_m \sin m\phi, \tag{5.46}$$

where $\sin m\phi$ are the half-range Fourier eigenfunctions and a_m are the respective half-range Fourier components.

(a) Orthogonality relation: The half-range Fourier eigenfunctions satisfy the orthogonality relation

$$\frac{2}{\pi}\int_0^\pi d\phi \, \sin m\phi \, \sin m'\phi = \delta_{mm'}. \tag{5.47}$$

(b) Fourier components: Using the orthogonality relations we can find the Fourier components to be

$$a_m = \frac{2}{\pi}\int_0^\pi d\phi \, \sin m\phi \, f(\phi). \tag{5.48}$$

(c) Completeness relation: The Fourier eigenfunctions satisfy the completeness relation

$$\frac{2}{\pi}\sum_{m=1}^{\infty} \sin m\phi \, \sin m\phi' = \delta(\phi - \phi'). \tag{5.49}$$

(d) Differential equation: The half-range Fourier eigenfunctions satisfy the differential equation

$$-\left[\frac{d^2}{d\phi^2} - m^2\right] \sin m\phi = 0. \tag{5.50}$$

Note that half-range Fourier eigenfunctions are zero at $\phi = 0$ and $\phi = \pi$.

5.4.1 Problems

1. (**20 points.**) Prove the orthogonality relation

$$\frac{2}{\pi}\int_0^\pi d\phi \, \sin m\phi \, \sin m'\phi = \delta_{mm'}. \tag{5.51}$$

Hint: Use exponential representation for sin functions.

2. (**20 points.**) Prove the completeness relation

$$\frac{2}{\pi}\sum_{m=1}^{\infty} \sin m\phi \, \sin m\phi' = \delta(\phi - \phi'). \tag{5.52}$$

Note that ϕ and ϕ' are limited to the range 0 to π.
Hint: Use exponential representation for sin functions.

3. (**20 points.**) For ϕ limited to the range

$$0 \leq \phi \leq \pi \tag{5.53}$$

show that $\cos\phi$ can be expressed as a linear combination of sin functions. That is,

$$\cos\phi = \sum_{m=1}^{\infty} a_m \sin m\phi. \tag{5.54}$$

Show that

$$a_m = \begin{cases} 0, & m = 1, 3, 5, \ldots, \\ \dfrac{4}{\pi}\dfrac{m}{(m^2-1)}, & m = 2, 4, 6, \ldots. \end{cases} \tag{5.55}$$

Note that the series expansion is not valid at the boundaries $\phi = 0$ and $\phi = \pi$.

4. (**20 points.**) For ϕ limited to the range
$$0 \leq \phi \leq \pi \tag{5.56}$$
show that 1 can be expressed as a linear combination of sin functions. That is,
$$1 = \sum_{m=1}^{\infty} a_m \sin m\phi. \tag{5.57}$$
Show that
$$a_m = \begin{cases} \frac{4}{\pi}\frac{1}{m}, & m = 1, 3, 5, \ldots, \\ 0, & m = 2, 4, 6, \ldots. \end{cases} \tag{5.58}$$
Note that the series expansion is not valid at the boundaries $\phi = 0$ and $\phi = \pi$. Evaluate the series at $\phi = \pi/2$ and find the series
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots. \tag{5.59}$$

5.5 Legendre polynomials

Refer EM notes.

Chapter 6

Linear differential equations

6.1 Wronskian

6.2 Initial conditions versus boundary conditions

6.3 Harmonic oscillator

6.4 Damped harmonic oscillator

1. (**Example.**) A damped harmonic oscillator, constituting of a body of mass m and a spring of spring constant k, is described by

$$ma = -kx - bv, \tag{6.1}$$

where x is position, $v = dx/dt$ is velocity, $a = dv/dt$ is acceleration, and b is the damping coefficient. Thus, we have the differential equation

$$\left[\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right] x(t) = 0 \tag{6.2}$$

with initial conditions

$$x(0) = x_0, \tag{6.3a}$$
$$\dot{x}(0) = v_0, \tag{6.3b}$$

where

$$\omega_0^2 = \frac{k}{m}, \qquad 2\gamma = \frac{b}{m}. \tag{6.4}$$

(a) $\gamma = 0$: In the absence of damping show that the solution is

$$x(t) = x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t. \tag{6.5}$$

(b) $\gamma < \omega_0$: Underdamped harmonic oscillator.

$$x(t) = e^{-\gamma t}\left[x_0 \cos\sqrt{\omega_0^2 - \gamma^2}\,t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}}\sin\sqrt{\omega_0^2 - \gamma^2}\,t\right]. \tag{6.6}$$

(c) $\gamma = \omega_0$: Critically damped harmonic oscillator.

$$x(t) = e^{-\omega_0 t}\left[x_0 + (v_0 + \omega_0 x_0)t\right]. \tag{6.7}$$

(d) $\gamma > \omega_0$: Overdamped harmonic oscillator.

$$x(t) = e^{-\gamma t}\left[x_0 \cosh\sqrt{\gamma^2 - \omega_0^2}\,t + \frac{(v_0 + \gamma x_0)}{\sqrt{\gamma^2 - \omega_0^2}} \sinh\sqrt{\gamma^2 - \omega_0^2}\,t\right]. \tag{6.8}$$

(e) Set $\omega_0 = 1$, which is equivalent to the substitution $\omega_0 t = \tau$, and sets the scale for the time t. That is, time is measured in units of $T = 2\pi/\omega_0$. The system is then completely characterized by the parameter γ/ω_0 and the initial conditions x_0 and v_0. Plot the solutions for the initial conditions $x_0 = 0$ and $v_0 = 1$.

2. (**20 points.**) Starting from the solution for the position of an underdamped harmonic oscillator ($\gamma < \omega_0$),

$$x(t) = e^{-\gamma t}\left[x_0 \cos\sqrt{\omega_0^2 - \gamma^2}\,t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin\sqrt{\omega_0^2 - \gamma^2}\,t\right], \tag{6.9}$$

obtain the solution for the velocity $v(t) = dx/dt$ of an underdamped harmonic oscillator ($\gamma < \omega_0$) in the form

$$v(t) = e^{-\gamma t}\left[v_0 \cos\sqrt{\omega_0^2 - \gamma^2}\,t - \frac{(\omega_0^2 x_0 + \gamma v_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin\sqrt{\omega_0^2 - \gamma^2}\,t\right]. \tag{6.10}$$

3. (**20 points.**) A critically damped harmonic oscillator is described by the differential equation

$$\left[\frac{d^2}{dt^2} + 2\omega_0 \frac{d}{dt} + \omega_0^2\right] x(t) = 0, \tag{6.11}$$

where ω_0 is a characteristic frequency. Find the solution $x(t)$ for initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. Plot $x(t)$ as a function of t in the following graph where $x_0\, e^{-\omega_0 t}$ is already plotted for reference. For what t is the solution $x(t)$ a maximum?

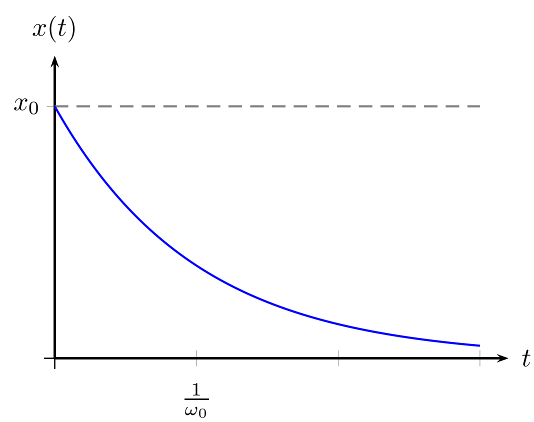

Figure 6.1: Critically damped harmonic oscillator.

4. (**20 points.**) A critically damped harmonic oscillator is described by the differential equation

$$\left[\frac{d^2}{dt^2} + 2\omega_0 \frac{d}{dt} + \omega_0^2\right] x(t) = 0, \tag{6.12}$$

where ω_0 is a characteristic frequency. Find the solution $x(t)$ for initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$. Plot $x(t)$ as a function of t in the graph in Figure 6.2, where ω_0 and v_0/ω_0 is used to set scales for time t and position $x(t)$. For what t is the solution $x(t)$ a maximum?

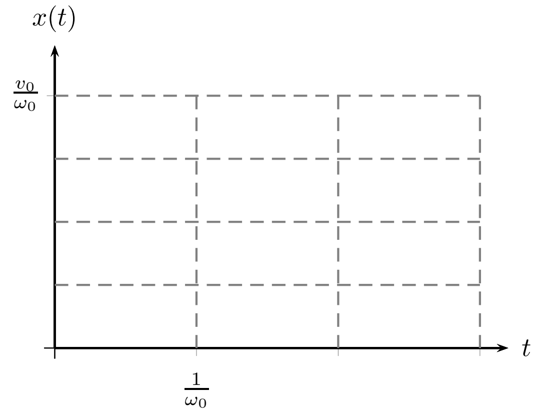

Figure 6.2: Critically damped harmonic oscillator.

5. (**20 points.**) Find the solution to the linear differential equation
$$\left[\frac{d^3}{dt^3} + 3\frac{d^2}{dt^2} + 3\frac{d}{dt} + 1\right] x(t) = 0 \tag{6.13}$$
for initial conditions $x(0) = 0$, $\dot{x}(0) = 0$, and $\ddot{x}(0) = a_0$.

6. (**20 points.**) A body experiencing only damping is described by the differential equation
$$\left[\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt}\right] x(t) = 0, \tag{6.14}$$
where γ is a measure of the damping. Find the solution $x(t)$ for initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$ to be
$$x(t) = x_0 + \frac{v_0}{2\gamma}\left[1 - e^{-2\gamma t}\right]. \tag{6.15}$$
Obtain the above expression starting from the solution for the overdamped harmonic oscillator ($\gamma > \omega_0$)
$$x(t) = e^{-\gamma t}\left[x_0 \cosh\sqrt{\gamma^2 - \omega_0^2}t + \frac{(v_0 + \gamma x_0)}{\sqrt{\gamma^2 - \omega_0^2}} \sinh\sqrt{\gamma^2 - \omega_0^2}t\right] \tag{6.16}$$
by setting $\omega_0 = 0$. Interpret the solution for $v_0 = 0$, why isn't there no motion?

7. (**20 points.**) Starting from the solution for the underdamped harmonic oscillator ($\gamma < \omega_0$),
$$x(t) = e^{-\gamma t}\left[x_0 \cos\sqrt{\omega_0^2 - \gamma^2}t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin\sqrt{\omega_0^2 - \gamma^2}t\right], \tag{6.17}$$
obtain the solution for the overdamped harmonic oscillator ($\gamma > \omega_0$),
$$x(t) = e^{-\gamma t}\left[x_0 \cosh\sqrt{\gamma^2 - \omega_0^2}t + \frac{(v_0 + \gamma x_0)}{\sqrt{\gamma^2 - \omega_0^2}} \sinh\sqrt{\gamma^2 - \omega_0^2}t\right]. \tag{6.18}$$

8. (**20 points.**) Starting from the solution for the underdamped harmonic oscillator ($\gamma < \omega_0$),
$$x(t) = e^{-\gamma t}\left[x_0 \cos\sqrt{\omega_0^2 - \gamma^2}t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin\sqrt{\omega_0^2 - \gamma^2}t\right], \tag{6.19}$$
obtain the solution for the critically damped harmonic oscillator ($\gamma = \omega_0$),
$$x(t) = e^{-\omega_0 t}\left[x_0 + (v_0 + \omega_0 x_0)t\right]. \tag{6.20}$$

9. (**20 points.**) The solution for the underdamped harmonic oscillator ($\gamma < \omega_0$) is

$$x(t) = e^{-\gamma t}\left[x_0 \cos\sqrt{\omega_0^2 - \gamma^2}\,t + \frac{(v_0 + \gamma x_0)}{\sqrt{\omega_0^2 - \gamma^2}} \sin\sqrt{\omega_0^2 - \gamma^2}\,t\right]. \tag{6.21}$$

For the initial condition $x_0 = 0$ we have

$$x(t) = \frac{v_0\, e^{-\gamma t}}{\sqrt{\omega_0^2 - \gamma^2}} \sin\sqrt{\omega_0^2 - \gamma^2}\,t. \tag{6.22}$$

Verify that the function

$$\frac{v_0\, e^{-\gamma t}}{\sqrt{\omega_0^2 - \gamma^2}} \tag{6.23}$$

is an envelope to the solution $x(t)$. Investigate if this is an envelope for the case $x_0 \neq 0$.

6.5 Forced harmonic oscillator

1. (**Example.**) A forced harmonic oscillator, in the absence of damping, constituting of a body of mass m and a spring of spring constant k, is described by

$$ma + kx = F(t), \tag{6.24}$$

where x is position, $v = dx/dt$ is velocity, $a = dv/dt$ is acceleration, and $F(t)$ is a driving force. Thus, we have the differential equation

$$-\left[\frac{d^2}{dt^2} + \omega_0^2\right] x(t) = A(t), \tag{6.25}$$

where

$$\omega_0^2 = \frac{k}{m}, \qquad A(t) = -\frac{F(t)}{m}. \tag{6.26}$$

Let us consider the case with initial conditions

$$x(0) = 0, \tag{6.27a}$$
$$\dot{x}(0) = 0. \tag{6.27b}$$

Verify by substitution that

$$x(t) = -\frac{1}{\omega_0}\int_0^t dt' \sin\omega_0(t - t') A(t') \tag{6.28}$$

is the solution.

2. (**Example.**) Consider the differential equation

$$-\left[\frac{d^2}{dt^2} + \omega_0^2\right] x(t) = -\omega_f^2 x_f \sin\omega_f t, \tag{6.29}$$

with initial conditions

$$x(0) = 0, \tag{6.30a}$$
$$\dot{x}(0) = 0. \tag{6.30b}$$

Verify by substitution that

$$x(t) = -x_f \frac{\omega_f^2}{(\omega_0^2 - \omega_f^2)} \frac{1}{\omega_0}\Big[\omega_f \sin\omega_0 t - \omega_0 \sin\omega_f t\Big] \tag{6.31}$$

is the solution. Show that the first term in the solution, called the transient solution, is solution to the homogeneous part of the differential equation. Show that the second term in the solution, called the steady-state solution, is a particular solution to the inhomogeneous differential equation.

Chapter 7

Partial differential equations

7.1 Vibrations in a string

1. (**20 points.**) Vibrations of a (guitar) string of length a are described by the height of oscillation

$$h = h(x,t) \tag{7.1}$$

that satisfies the differential equation

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 h}{\partial t^2} \tag{7.2}$$

with boundary conditions

$$h(0,t) = 0, \tag{7.3a}$$
$$h(a,t) = 0, \tag{7.3b}$$

and initial conditions

$$h(x,0) = h_0(x), \tag{7.4a}$$
$$\left\{\frac{\partial}{\partial t}h(x,t)\right\}_{t=0} = 0. \tag{7.4b}$$

Here v is the speed of propagation given in terms of the tension T in the string (presumed to be uniform) and mass per unit length λ of the string, $v = \sqrt{T/\lambda}$. The given function $h_0(x)$ characterizes how the string is released initially.

(a) Let $F(x)$ and $T(t)$ be eigenfunctions in terms of which the solution $h(x,t)$ can be described. Thus, the product

$$F(x)T(t) \tag{7.5}$$

satisfies the differential equation for $h(x,t)$. Substitute in Eq. (7.2) and rearrange to obtain

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{T(t)}\frac{1}{v^2}\frac{\partial^2 T(t)}{\partial t^2}. \tag{7.6}$$

(b) The left hand side of Eq. (7.6) is only dependent on x and the right hand side is only dependent on t. Argue that this can be satisfied for arbitrary x and t only if each side is equal to the same constant, say α. Note that α could be complex. This is called separation of variables. Thus, we have

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = \alpha = \frac{1}{T(t)}\frac{1}{v^2}\frac{\partial^2 T(t)}{\partial t^2}. \tag{7.7}$$

(c) Rewrite the equation of $X(x)$ in the form

$$\frac{\partial^2 X}{\partial x^2} = \alpha X. \tag{7.8}$$

Verify that it permits the solution

$$X(x) = Ae^{\sqrt{\alpha}x} + Be^{-\sqrt{\alpha}x}. \tag{7.9}$$

Show that the boundary conditions in Eq. (7.3) impose the conditions

$$A + B = 0, \tag{7.10a}$$

$$Ae^{\sqrt{\alpha}L} + Be^{-\sqrt{\alpha}L} = 0. \tag{7.10b}$$

Verify that $A = 0$ and $B = 0$ is a solution. However, it is a trivial solution, because it corresponds to no motion. Argue that Eq. (7.10) is also satisfied if

$$\det \begin{pmatrix} 1 & 1 \\ e^{\sqrt{\alpha}a} & e^{-\sqrt{\alpha}a} \end{pmatrix} = 0. \tag{7.11}$$

Thus, derive

$$\alpha = -m^2 \frac{\pi^2}{a^2}, \qquad m = 0, \pm 1, \pm 2, \ldots. \tag{7.12}$$

Thus, conclude that $X(x)$ satisfies solutions of the form

$$X(x) = Ae^{im\pi\frac{x}{a}} + Be^{-im\pi\frac{x}{a}}. \tag{7.13}$$

Requiring this solution to satisfy the boundary conditions show that

$$X(x) = A' \sin\left(m\pi \frac{x}{a}\right), \tag{7.14}$$

where $A' = 2iA$. Observe that the boundary conditions do not determine A', it is left arbitrary.

(d) Use the Wronskian to show that the eigenfunctions

$$\sin\left(m\pi \frac{x}{a}\right), \qquad m = 1, 2, 3, \ldots, \tag{7.15}$$

constitute linearly independent solutions. Verify that these functions satisfy the orthogonality relations

$$\frac{2}{a}\int_0^a dx \, \sin\left(m\pi \frac{x}{a}\right) \sin\left(m'\pi \frac{x}{a}\right) = \delta_{mm'}. \tag{7.16}$$

These functions also satisfy the completeness relation

$$\frac{2}{a}\sum_{m=1}^{\infty} dx \, \sin\left(m\pi \frac{x}{a}\right) \sin\left(m\pi \frac{x'}{a}\right) = \delta(x - x'), \tag{7.17}$$

which need not be proved here. This allows us to expand the desired solution $h(x,t)$ in terms of these eigenfunctions as

$$h(x,t) = \sum_{m=1}^{\infty} T_m(t) \sin\left(m\pi \frac{x}{a}\right), \tag{7.18}$$

where $T_m(t)$ are the respective components. Verify that $h(x,t)$ satisfies the boundary conditions.

(e) Substituting this in the original differential equation show that

$$\sum_{m=1}^{\infty} \sin\left(m\pi\frac{x}{a}\right)\left[\frac{\partial^2 T_m}{\partial t^2} + \left(m\pi\frac{v}{a}\right)^2 T_m\right] = 0. \tag{7.19}$$

Using the completeness relation deduce the differential equations

$$\frac{\partial^2 T_m}{\partial t^2} = -\left(m\pi\frac{v}{a}\right)^2 T_m, \tag{7.20}$$

for each m. The solutions for these equations are of the form

$$T_m(t) = C_m \sin\left(m\pi\frac{v}{a}t\right) + D_m \cos\left(m\pi\frac{v}{a}t\right). \tag{7.21}$$

Thus, show that

$$h(x,t) = \sum_{m=1}^{\infty}\left[C_m \sin\left(m\pi\frac{v}{a}t\right) + D_m \cos\left(m\pi\frac{v}{a}t\right)\right]\sin\left(m\pi\frac{x}{a}\right). \tag{7.22}$$

Using the initial conditions show that

$$h_0(x) = \sum_{m=1}^{\infty} D_m \sin\left(m\pi\frac{x}{a}\right), \tag{7.23a}$$

$$0 = \sum_{m=1}^{\infty} C_m \left(m\pi\frac{v}{a}\right)\sin\left(m\pi\frac{x}{a}\right). \tag{7.23b}$$

Thus, learn that

$$C_m = 0. \tag{7.24}$$

Using orthogonality relations invert Eq. (7.23a) to derive

$$D_m = \frac{2}{a}\int_0^a dx\, h_0(x)\sin\left(m\pi\frac{x}{a}\right). \tag{7.25}$$

(f) Together, summarize the solution to be

$$h(x,t) = \sum_{m=1}^{\infty} D_m \cos\left(m\pi\frac{v}{a}t\right)\sin\left(m\pi\frac{x}{a}\right), \tag{7.26}$$

where D_m is determined using the initial condition $h_0(x)$ using

$$D_m = \frac{2}{a}\int_0^a dx\, h_0(x)\sin\left(m\pi\frac{x}{a}\right). \tag{7.27}$$

Find all D_m's for

$$h_0(x) = H\sin\left(\pi\frac{x}{a}\right). \tag{7.28}$$

Hint: Use the orthogonality relations.

Chapter 8

Fluid dynamics

8.1 Navier-Stokes equations

Continuity equation.

Momentum conservation.

Energy conservation.

8.2 Low-Reynolds-number flow (Incomplete)

8.2.1 Stokes equations

When the inertial effects are negligible the Navier-Stokes equations for the flow velocity $\mathbf{u}(\mathbf{r})$ and the fluid pressure $p(\mathbf{r})$ takes the form

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{8.1}$$

$$\boldsymbol{\nabla} \cdot \Big[p\mathbf{1} - \mu \boldsymbol{\nabla} \mathbf{u} \Big] = \mathbf{F}, \tag{8.2}$$

where μ is the uniform viscosity coefficient and $\mathbf{f}(\mathbf{r})$ is an external force density function. The boundary conditions on the flow are imposed at $|\mathbf{r}| \to \infty$ to be $\mathbf{u} \to 0$ and $p \to p_\infty$.

Notes:

1. Solenoidal forcing function:

$$\boldsymbol{\nabla} \cdot \mathbf{F} = 0, \tag{8.3}$$

which allows the solution

$$\mathbf{f} = \mu \boldsymbol{\nabla} \times \boldsymbol{\omega}, \tag{8.4}$$

where $\boldsymbol{\omega}$ is the vorticity.

2. Vorticity:

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}. \tag{8.5}$$

3. Stream function

$$\mathbf{u} = \boldsymbol{\nabla} \times \boldsymbol{\psi}. \tag{8.6}$$

4. Derive Stokes law for the drag force on a sphere.

8.2.2 Problems

1. (**20 points.**) Read the article titled 'Life at low Reynolds number' by E. M. Purcell, American Journal of Physics 45 (1977) 3. Here is the link to the article:

   http://dx.doi.org/10.1119/1.10903

   Here is a question asked to verify the understanding of the concept being discussed in the paper. Imagine a micrometer sized bacteria, shaped like a human, swimming in water using the methods used by a typical human swimmer. Qualitatively describe the motion of this hypothetical bacteria.

Chapter 9

Green's function

Refer EM notes.

Chapter 10

Legnedre polynomials

10.1 Dipole moment

1. (**30 points.**) (Based on Griffiths 3rd/4th ed., Problem 4.9.)

   (a) The electric field of a point charge q at distance $\mathbf{r}$ is
   $$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}. \tag{10.1}$$
   The force on a point dipole in the presence of an electric field is
   $$\mathbf{F} = (\mathbf{d}\cdot\boldsymbol{\nabla})\mathbf{E}. \tag{10.2}$$
   Use these to find the force on a point dipole due to a point charge.

   (b) The electric field of a point dipole $\mathbf{d}$ at distance $\mathbf{r}$ from the dipole is given by
   $$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3\,\hat{\mathbf{r}}\,(\mathbf{d}\cdot\hat{\mathbf{r}}) - \mathbf{d}\big]. \tag{10.3}$$
   The force on a point charge in the presence of an electric field is
   $$\mathbf{F} = q\mathbf{E}. \tag{10.4}$$
   Use these to find the force on a point charge due to a point dipole.

   (c) Confirm that above two forces are equal in magnitude and opposite in direction, as per Newton's third law.

2. (**40 points.**) (Based on Griffiths 3rd/4th ed., Problem 4.8.)
   We showed in class that the electric field of a point dipole $\mathbf{d}$ at distance $\mathbf{r}$ from the dipole is given by the expression
   $$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3\,\hat{\mathbf{r}}\,(\mathbf{d}\cdot\hat{\mathbf{r}}) - \mathbf{d}\big]. \tag{10.5}$$
   The interaction energy of a point dipole $\mathbf{d}$ in the presence of an electric field is given by
   $$U = -\mathbf{d}\cdot\mathbf{E}. \tag{10.6}$$
   Further, the force between the two dipoles is given by
   $$\mathbf{F} = -\boldsymbol{\nabla}U. \tag{10.7}$$
   Use these expressions to derive

(a) the interaction energy between two point dipoles separated by distance $\mathbf{r}$ to be

$$U = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[\mathbf{d}_1\cdot\mathbf{d}_2 - 3\,(\mathbf{d}_1\cdot\hat{\mathbf{r}})(\mathbf{d}_2\cdot\hat{\mathbf{r}})\big]. \tag{10.8}$$

(b) the force between the two dipoles to be

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{3}{r^4}\big[(\mathbf{d}_1\cdot\mathbf{d}_2)\,\hat{\mathbf{r}} + (\mathbf{d}_1\cdot\hat{\mathbf{r}})\,\mathbf{d}_2 + (\mathbf{d}_2\cdot\hat{\mathbf{r}})\,\mathbf{d}_1 - 5\,(\mathbf{d}_1\cdot\hat{\mathbf{r}})(\mathbf{d}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\big]. \tag{10.9}$$

(c) Are the forces central? That is, is the force in the direction of $\mathbf{r}$?

(d) Are the forces on the dipole equal in magnitude and opposite in direction? That is, do they satisfy Newton's third law?

3. (**20 points.**) For what a, b, and $\mathbf{c}$, is the relation

$$\boldsymbol{\nabla}\left[\frac{(\mathbf{d}_1\cdot\hat{\mathbf{r}})(\mathbf{d}_2\cdot\hat{\mathbf{r}})}{r^3}\right] = \frac{a\,(\mathbf{d}_1\cdot\hat{\mathbf{r}})\,\mathbf{d}_2 + b\,(\mathbf{d}_2\cdot\hat{\mathbf{r}})\,\mathbf{d}_1 + (\mathbf{d}_1\cdot\hat{\mathbf{r}})(\mathbf{d}_2\cdot\hat{\mathbf{r}})\,\mathbf{c}}{r^4} \tag{10.10}$$

an identity. What are the dimensions of a, b, and $\mathbf{c}$?

4. (**20 points.**) The potential energy of an electric dipole $\mathbf{p}$ in an electric field, that is not necessarily uniform, is

$$U = -\mathbf{p}\cdot\mathbf{E}. \tag{10.11}$$

Restricting to electrostatics, ($\boldsymbol{\nabla}\cdot\mathbf{D} = \rho$ and $\boldsymbol{\nabla}\times\mathbf{E} = 0$,) show that the force on the electric dipole moment

$$\mathbf{F} = -\boldsymbol{\nabla}U \tag{10.12}$$

is given in terms of the directional derivative of the electric field in the direction of the electric dipole moment,

$$\mathbf{F} = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}. \tag{10.13}$$

5. (**10 points.**) Interaction energy of a dipole $\mathbf{d}$ with an electric field $\mathbf{E}$ is

$$U = -\mathbf{d}\cdot\mathbf{E} = -dE\cos\theta. \tag{10.14}$$

The torque on the dipole due to the electric field is

$$\boldsymbol{\tau} = \mathbf{d}\times\mathbf{E}. \tag{10.15}$$

Force is a manifestation of the systems tendency to minimize its energy, and in this spirit torque is defined as,

$$\tau = -\frac{\partial}{\partial\theta}U = -dE\sin\theta. \tag{10.16}$$

Show that there is no inconsistency, in sign, between the two definitions of torque.

6. (**10 points.**) Show that the effective charge density, ρ_{eff}, and the effective current density, $\mathbf{j}_{\text{eff}}$,

$$\rho_{\text{eff}} = -\boldsymbol{\nabla}\cdot\mathbf{P}, \tag{10.17}$$

$$\mathbf{j}_{\text{eff}} = \frac{\partial}{\partial t}\mathbf{P} + \boldsymbol{\nabla}\times\mathbf{M}, \tag{10.18}$$

satisfy the equation of charge conservation

$$\frac{\partial}{\partial t}\rho_{\text{eff}} + \boldsymbol{\nabla}\cdot\mathbf{j}_{\text{eff}} = 0. \tag{10.19}$$

7. (**10 points.**) The magnetic dipole moment of charge q_a moving with velocity $\mathbf{v}_a$ is

$$\boldsymbol{\mu} = \frac{1}{2} q_a \mathbf{r}_a \times \mathbf{v}_a, \tag{10.20}$$

where $\mathbf{r}_a$ is the position of the charge. For a charge moving along a circular orbit of radius r_a, with constant speed v_a, deduce the magnetic moment

$$\boldsymbol{\mu} = IA\,\hat{\mathbf{n}}, \qquad I = \frac{q_a}{\Delta t}\frac{v_a \Delta t}{2\pi r_a} \qquad A = \pi r_a^2, \tag{10.21}$$

where $\hat{\mathbf{n}}$ points along $\mathbf{r}_a \times \mathbf{v}_a$.

8. (**30 points.**) Identify the orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ in the expression for magnetic dipole moment, then generalize to total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$, where $\mathbf{S}$ is the spin of the particle. Thus, deduce the relation

$$\boldsymbol{\mu} = \gamma \mathbf{J}, \tag{10.22}$$

where γ is the gyromagnetic ratio of a particle. A magnetic dipole moment feels a torque given by

$$\boldsymbol{\tau} = \frac{d\mathbf{J}}{dt} = \boldsymbol{\mu} \times \mathbf{B}, \tag{10.23}$$

which causes the magnetic moment to precess around the magnetic field. Solve the above equations and find the precession angular frequency in terms of γ and B.

9. (**30 points.**) Consider a circular loop of wire carrying current I whose magnetic moment is given by $\boldsymbol{\mu} = IA\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ points perpendicular to the plane containing the loop (satisfying the right hand sense) and A is the area of the loop. Consider the case $\hat{\mathbf{n}} = \hat{\mathbf{x}}$. What is the magnitude and direction of the torque experienced by this loop in the presence of a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{y}}$. Describe the resultant motion of the loop. (Hint: The torque experienced by a magnetic moment $\boldsymbol{\mu}$ in a magnetic field $\mathbf{B}$ is $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$.)

10.2 Legendre polynomials

1. (**Recurrence relation.**) The Legendre polynomials $P_l(x)$ of degree l are defined, or generated, by expanding the electric (or gravitational) potential of a point charge,

$$\frac{\alpha}{|\mathbf{r} - \mathbf{r}'|} = \frac{\alpha}{r_>} \frac{1}{\sqrt{1 + \left(\frac{r_<}{r_>}\right)^2 - 2\left(\frac{r_<}{r_>}\right)\cos\gamma}} = \frac{\alpha}{r_>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>}\right)^l P_l(\cos\gamma), \tag{10.24}$$

where

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'), \tag{10.25}$$

and

$$r_< = \text{Minimum}(r, r'), \tag{10.26a}$$
$$r_> = \text{Maximum}(r, r'). \tag{10.26b}$$

Thus, in terms of variables

$$t = \frac{r_<}{r_>}, \qquad 0 \leq t < \infty, \tag{10.27}$$

and

$$x = \cos\gamma, \qquad -1 \leq x < 1, \tag{10.28}$$

we can define the generating function for the Legendre polynomials as

$$g(t,x) = \frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{10.29}$$

Setting $t = 0$ in the above relation we immediately learn that

$$P_0(x) = 1. \tag{10.30}$$

Legenendre polynomials of higher degrees can be derived by Taylor expansion of the generating function. However, for large degrees it is more efficient to derive a recurrence relation. To derive the recurrence relation for Legendre polynomials we begin by differentiating the generating function with respect to t to obtain

$$\frac{\partial g}{\partial t} = \frac{(x-t)}{(1+t^2-2xt)^{\frac{3}{2}}} = \sum_{l=1}^{\infty} l\, t^{l-1} P_l(x). \tag{10.31}$$

Inquire why the sum on the right hand side now starts from $l = 1$. The second equality can be rewritten in the form

$$\frac{(x-t)}{\sqrt{1+t^2-2xt}} = (1+t^2-2xt) \sum_{l=1}^{\infty} l\, t^{l-1} P_l(x), \tag{10.32}$$

and implies

$$(x-t) \sum_{l=0}^{\infty} t^l P_l(x) = (1+t^2-2xt) \sum_{l=1}^{\infty} l\, t^{l-1} P_l(x). \tag{10.33}$$

Express this in the form

$$\begin{aligned} t^0 \Big[xP_0(x) - P_1(x) \Big] \quad &+ t^1 \Big[3xP_1(x) - P_0(x) - 2P_2(x) \Big] \\ &+ \sum_{l=2}^{\infty} t^l \Big[(2l+1)x\, P_l(x) - l\, P_{l-1}(x) - (l+1)\, P_{l+1}(x) \Big] = 0. \end{aligned} \tag{10.34}$$

Thus, using the completeness property of Taylor expansion, that is, equating the coefficients of powers of t in the expansion, we have, for t^0 and t^1,

$$P_1(x) = xP_0(x), \tag{10.35a}$$

$$2P_2(x) = 3xP_1(x) - P_0(x), \tag{10.35b}$$

and matching powers of t^l for $l \geq 2$ we obtain the recurrence relation for Legendre polynomials as

$$(l+1)\, P_{l+1}(x) = (2l+1)x\, P_l(x) - l\, P_{l-1}(x), \qquad l = 0, 1, 2, 3, \ldots. \tag{10.36}$$

Note that the recurrence relations in Eq. (10.39), for $l = 0$ and $l = 1$, reproduces Eqs. (10.35). The recurrence relations in Eq. (10.39) can be reexpressed in the form

$$l\, P_l(x) = (2l-1)x\, P_{l-1}(x) - (l-1)\, P_{l-2}(x), \qquad l = 1, 2, 3, \ldots. \tag{10.37}$$

Thus, Eq. (10.37) generates Legendre polynomials of all degrees starting from $P_0(x) = 1$, which was obtained in Eq. (10.30).

2. (**Differential equation.**) The generating function for the Legendre polynomials $P_l(x)$ of degree l is

$$g(t,x) = \frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{10.38}$$

(a) Starting from the generating function and differentiating with respect to t we derived the recurrence relation for Legendre polynomials in Eq. (10.39),

$$(l+1)\,P_{l+1}(x) = (2l+1)x\,P_l(x) - l\,P_{l-1}(x), \qquad l = 0,1,2,\ldots, \tag{10.39}$$

in terms of

$$P_0(x) = 1 = g(0,x). \tag{10.40}$$

Differentiating the recurrence relation with respect to x show that

$$(2l+1)\,P_l + (2l+1)x\,P_l' = l\,P_{l-1}' + (l+1)\,P_{l+1}', \qquad l = 0,1,2,\ldots, \tag{10.41}$$

where we supressed the dependence in x and prime in the superscript of $P_l'(x)$ denotes derivative with respect to the argument x.

(b) Differentiating the generating function with respect to x show that

$$\frac{\partial g}{\partial x} = \frac{t}{(1+t^2-2xt)^{\frac{3}{2}}} = \sum_{l=0}^{\infty} t^l P_l'(x). \tag{10.42}$$

Show that the second equality can be rewritten in the form

$$\frac{t}{\sqrt{1+t^2-2xt}} = (1+t^2-2xt)\sum_{l=0}^{\infty} t^l P_l'(x), \tag{10.43}$$

and implies

$$t\sum_{l=0}^{\infty} t^l P_l(x) = (1+t^2-2xt)\sum_{l=0}^{\infty} t^l P_l'(x). \tag{10.44}$$

Express this in the form

$$t^0\Big[P_0'(x)\Big] \quad + t^1\Big[P_1'(x) - 2xP_0'(x) - P_0(x)\Big] + \sum_{l=2}^{\infty} t^l\Big[P_l'(x) + P_{l-2}'(x) - 2x\,P_{l-1}'(x) - P_{l-1}(x)\Big] = 0. \tag{10.45}$$

Then, using the completeness property of Taylor expansion, that is, equating the coefficients of powers of t in the expansion, show that, for t^0 and t^1,

$$P_0'(x) \;=\; 0, \tag{10.46a}$$
$$P_1'(x) \;=\; P_0(x) = 1, \tag{10.46b}$$

and matching powers of t^l for $l \geq 2$ derive a recurrence relation for the derivative of Legendre polynomials as

$$2x\,P_{l-1}' + P_{l-1} = P_l' + P_{l-2}', \qquad l = 2,3,\ldots. \tag{10.47}$$

Here, we shall find it convenient to use the above recurrence relations in the form

$$2x\,P_l' + P_l = P_{l+1}' + P_{l-1}', \qquad l = 1,2,3,\ldots, \tag{10.48}$$

which is obtained by setting $l \to l+1$.

(c) Equations (10.41) and (10.48) are linear set of equations for P_{l-1}' and P_{l+1}' in terms of P_l and P_l'. Solve them to find

$$P_{l+1}' = xP_l' + (l+1)\,P_l, \qquad l = 0,1,2,\ldots, \tag{10.49a}$$
$$P_{l-1}' = x\,P_l' - l\,P_l. \qquad l = 1,2,3,\ldots. \tag{10.49b}$$

(d) Using $l \to l-1$ in Eq. (10.49a) show that

$$P_l' = x\, P_{l-1}' + l\, P_{l-1}. \tag{10.50}$$

Then, substitute Eq. (10.49b) to obtain

$$(1-x^2)\, P_l' = l\, P_{l-1} - xl\, P_l. \tag{10.51}$$

Differentiate the above equation and substitute Eq. (10.49b) again to derive the differential equation for Legendre polynomials as

$$\left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x} + l(l+1)\right] P_l(x) = 0. \tag{10.52}$$

Substitute $x = \cos\theta$ to rewrite the differential equation in the form

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + l(l+1)\right] P_l(\cos\theta) = 0. \tag{10.53}$$

3. (**Rodrigues formula for Legendre polynomials.**)
The generating function for the Legendre polynomials $P_l(x)$ of degree l is

$$g(t,x) = \frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{10.54}$$

(a) Using binomial expansion show that

$$\frac{1}{\sqrt{1-y}} = \sum_{m=0}^{\infty} y^m \frac{(2m)!}{\big[m!\, 2^m\big]^2} \tag{10.55}$$

and

$$(2xt-t^2)^m = \sum_{n=0}^{\infty} \frac{m!}{n!(m-n)!}(2xt)^{m-n} t^{2n} (-1)^n. \tag{10.56}$$

Thus, show that

$$\frac{1}{\sqrt{1+t^2-2xt}} = \sum_{m=0}^{\infty}\sum_{n=0}^{m} t^{m+n} \frac{(2m)!}{m!n!(m-n)!2^{m+n}} x^{m-n}(-1)^n. \tag{10.57}$$

(b) In Figure 10.1 we illustrate how we change the double sum in m and n to variables l and s. This is achieved using the substitutions

$$m+n = l, \tag{10.58a}$$
$$m-n = 2s, \tag{10.58b}$$

which corresponds to

$$2m = l+2s, \qquad m = \frac{l}{2}+s, \quad \text{and} \quad n = \frac{l}{2}-s. \tag{10.59}$$

The counting on the variable s, for given l, follows the pattern,

$$l \text{ even}: \quad 2s = 0, 2, 4, \ldots, l, \tag{10.60a}$$
$$l \text{ odd}: \quad 2s = 1, 3, 5, \ldots, l. \tag{10.60b}$$

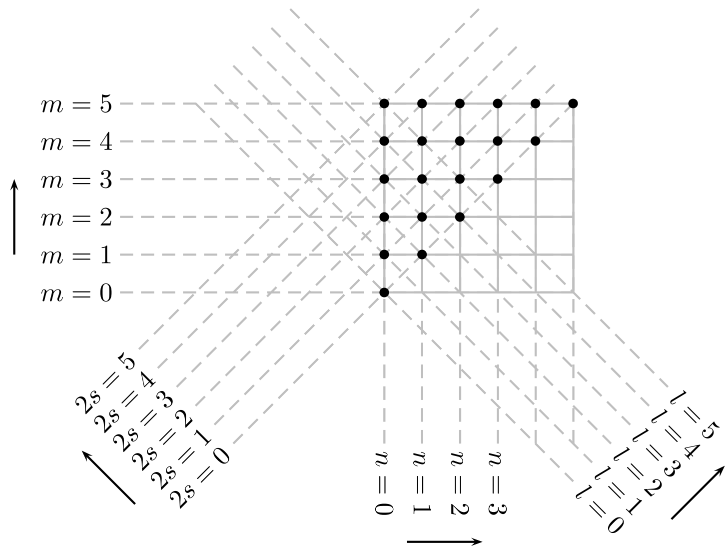

Figure 10.1: Double summation.

Show that in terms of l and s the double summation can be expressed as

$$\frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty}\sum_{s} t^l \frac{(l+2s)!}{\left(\frac{l}{2}+s\right)!\left(\frac{l}{2}-s\right)!(2s)!2^l} x^{2s}(-1)^{\frac{l}{2}-s}, \tag{10.61}$$

where the limits on the sum in s are dictated by Eqs. (10.60) depending on l being even or odd. Thus, read out the polynomial expression for Legendre polynomials of degree l to be

$$P_l(x) = \sum_{s} \frac{(l+2s)!}{\left(\frac{l}{2}+s\right)!\left(\frac{l}{2}-s\right)!(2s)!2^l} x^{2s}(-1)^{\frac{l}{2}-s}, \tag{10.62}$$

where the summation on s depends on whether l is even or odd.

(c) Show that

$$\left(\frac{d}{dx}\right)^l x^{l+2s} = \frac{(l+2s)!}{(2s)!} x^{2s}. \tag{10.63}$$

Thus, show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{s} \frac{l!}{\left(\frac{l}{2}+s\right)!\left(\frac{l}{2}-s\right)!} x^{l+2s}(-1)^{\frac{l}{2}-s}. \tag{10.64}$$

(d) For even l the summation in s runs from $s=0$ to $s=l/2$, Thus, writing $l+2s = 2[l-\left(\frac{l}{2}-s\right)]$, show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{s=0}^{\frac{l}{2}} \frac{l!}{\left(\frac{l}{2}+s\right)!\left(\frac{l}{2}-s\right)!} (x^2)^{l-\left(\frac{l}{2}-s\right)}(-1)^{\left(\frac{l}{2}-s\right)}. \tag{10.65}$$

Then, substituting

$$\frac{l}{2} - s = n, \tag{10.66}$$

show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{n=0}^{\frac{l}{2}} \frac{l!}{(l-n)!n!} (x^2)^{l-n}(-1)^n. \tag{10.67}$$

Note that the summation on n runs from $n = 0$ to $n = l/2$. If we were to extend this sum to $n = l$ verify that the additional terms will have powers in x less than l. Since the terms in the sum are acted upon by l derivatives with respect to x these additional terms will not contribute. Thus, show that

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{l!\,2^l}. \tag{10.68}$$

Similarly, for odd l the summation is s runs as

$$2s = 1, 3, 5, \ldots, l, \tag{10.69}$$

or

$$\frac{2s-1}{2} = 0, 1, 2, \ldots, \frac{l-1}{2}. \tag{10.70}$$

Thus, substituting

$$s' = \frac{2s-1}{2} = s - \frac{1}{2}, \tag{10.71}$$

show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{s=0}^{\frac{l-1}{2}} \frac{l!}{\left(\frac{l+1}{2}+s\right)!\left(\frac{l-1}{2}-s\right)!} x^{l+1+2s}(-1)^{\left(\frac{l-1}{2}-s\right)}. \tag{10.72}$$

Substituting

$$\frac{l-1}{2} - s = n \tag{10.73}$$

and writing

$$\frac{l+1}{2} + s = l - \left(\frac{l-1}{2} - s\right) \tag{10.74}$$

show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{n=0}^{\frac{l-1}{2}} \frac{l!}{(l-n)!n!}(x^2)^{l-n}(-1)^n. \tag{10.75}$$

Again, like in the case of even l we can extend the sum on n beyond $n = (l-1)/2$, because they do not survive under the action of l derivatives with respect to x. Thus, again, we have

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{l!\,2^l}, \tag{10.76}$$

which is exactly the form obtained for even l. The expression in Eq. (10.76) is the Rodrigues formula for generating the Legendre polynomials of degree l.

4. (**20 points.**) (**Orthogonality relations**.)
   Refer 2022Nov28.

10.2.1 Problems

1. (**20 points.**) Using Mathematica (or another graphing tool) plot the Legendre polynomials $P_l(x)$ for $l = 0, 1, 2, 3, 4$ on the same plot. Note that $-1 \leq x \leq 1$. Based on the pattern you see what can you conclude about the number of roots for $P_l(x)$. In Mathematica these plots are generated using the following commands:
   `Plot[{LegendreP[0,x], LegendreP[1,x], LegendreP[2,x], LegendreP[3,x],`
   `LegendreP[4,x] },{x,-1,1}]`
   Compare your plots with those in Wikipedia article on 'Legendre Polynomials'. While there read the Wikipedia article on Adrien-Marie Legendre and the associated 'Portrait Debacle'.

2. (**20 points.**) Legendre polynomials are conveniently generated using the relation

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}, \tag{10.77}$$

where $-1 \leq x \leq 1$. Evaluate Legendre polynomials of degree $l = 0, 1, 2, 3, 4$ in this manner.

3. (**20 points.**) Legendre polynomials $P_l(x)$ satisfy the relation

$$\int_{-1}^{1} dx\, P_l(x) = 0 \quad \text{for} \quad l \geq 1. \tag{10.78}$$

Verify this explicitly for $l = 0, 1, 2, 3, 4$.

4. (**20 points.**) Legendre polynomials satisfy the differential equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + l(l+1)\right] P_l(\cos\theta) = 0. \tag{10.79}$$

Verify this explicitly for $l = 0, 1, 2, 3, 4$.

5. (**20 points.**) Legendre polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}. \tag{10.80}$$

Verify this explicitly for $l = 0, 1, 2$ and $l' = 0, 1, 2$. The orthogonality relation is also expressed as

$$\int_{0}^{\pi} \sin\theta d\theta\, P_l(\cos\theta) P_{l'}(\cos\theta) = \frac{2}{2l+1}\delta_{ll'}. \tag{10.81}$$

6. (**20 points.**) Legendre polynomials satisfy the completeness relation

$$\sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(x) P_l(x') = \delta(x - x'). \tag{10.82}$$

This is for your information. No work needed. The completeness relation is also expressed as

$$\sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(\cos\theta) P_l(\cos\theta') = \frac{\delta(\theta - \theta')}{\sin\theta}. \tag{10.83}$$

7. (**Example.**) The Legendre polynomials of order l are

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}. \tag{10.84}$$

In particular,

$$P_0(x) = 1, \tag{10.85a}$$
$$P_1(x) = x, \tag{10.85b}$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}. \tag{10.85c}$$

The expansion

$$F(x,t) = \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{l=0}^{\infty} t^l P_l(x), \qquad |t| < 1, \tag{10.86}$$

is usually referred to as the generating function for Legendre's polynomials. From it all the properties of these polynomials may be derived.

8. (**Example.**) The Legendre polynomials of order l satisfy the recurrence relation

$$(2l+1)xP_l(x) = (l+1)P_{l+1}(x) + lP_{l-1}(x), \qquad l = 1, 2, 3, \ldots. \tag{10.87}$$

Recall,

$$P_0(x) = 1, \tag{10.88a}$$
$$P_1(x) = x. \tag{10.88b}$$

Derive the explicit expression for $P_4(x)$ using the recurrence relation.

9. (**20 points.**) Express the function

$$\sigma(\theta) = \cos^2\theta \tag{10.89}$$

in terms of Legendre polynomials.
Solution:

$$\sigma(\theta) = \frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_0(\cos\theta). \tag{10.90}$$

10. (**20 points.**) Express the function

$$\sigma(\theta) = \cos 2\theta \tag{10.91}$$

in terms of Legendre polynomials.
Solution:

$$\sigma(\theta) = \frac{4}{3}P_2(\cos\theta) - \frac{1}{3}P_0(\cos\theta). \tag{10.92}$$

11. (**20 points.**) Legendre polynomials satisfy the completeness relation

$$\sum_{l=0}^{n} P_l(\cos\theta)P_{n-l}(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{10.93}$$

Verify this explicitly for $l = 0, 1, 2$. Prove this for arbitrary n. No work needed. I have still not attempted on it.

12. (**20 points.**) The generating function for the Legendre polynomials $P_l(x)$ of degree l is

$$g(t,x) = \frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{10.94}$$

Evaluate $P_{11}(0)$ and $P_{12}(0)$.

10.3 Electric potential of $2l$-pole

1. (**10 points.**) The surface charge density on the surface of a charged sphere is given by

$$\sigma(\theta,\phi) = \frac{Q}{4\pi a^2}\cos^2\theta, \tag{10.95}$$

where θ is the polar angle in spherical coordinates. Express this charge distribution in terms of the Legendre polynomials. Recall,

$$P_0(\cos\theta) = 1, \tag{10.96a}$$
$$P_1(\cos\theta) = \cos\theta, \tag{10.96b}$$
$$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}. \tag{10.96c}$$

2. (**10 points.**) The induced charge on the surface of a spherical conducting shell of radius a due to a point charge q placed a distance b away from the center is given by

$$\rho(\mathbf{r}) = \sigma(\theta,\phi)\,\delta(r-a), \tag{10.97}$$

where

$$\sigma(\theta,\phi) = -\frac{q}{4\pi a}\frac{(r_>^2 - r_<^2)}{(a^2+b^2-2ab\cos\theta)^{\frac{3}{2}}}, \tag{10.98}$$

where $r_< = \text{Min}(a,b)$ and $r_> = \text{Max}(a,b)$. Calculate the dipole moment of this charge configuration (excluding the original charge q) using

$$\mathbf{d} = \int d^3r\,\mathbf{r}\,\rho(\mathbf{r}), \tag{10.99}$$

for the two cases $a < b$ and $a > b$, representing the charge being inside or outside the sphere. (Hint: First complete the r integral and the ϕ integral. Then, for the θ integral substitute $a^2 + b^2 - 2ab\cos\theta = y$.)

3. (**20 points.**) Consider the electric potential due to a solid sphere with uniform charge density Q. The angular integral in this evaluation involves the integral

$$\frac{1}{2}\int_{-1}^{1} dt \frac{1}{\sqrt{r^2 + r'^2 - 2rr't}}. \tag{10.100}$$

Evaluate the integral for $r < r'$ and $r' < r$, where r and r' are distances measured from the center of the sphere. (Hint: Substitute $r^2 + r'^2 - 2rr't = y$.)

4. (**20 points.**) Recollect Legendre polynomials of order l

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}. \tag{10.101}$$

In particular

$$P_0(x) = 1, \tag{10.102a}$$
$$P_1(x) = x, \tag{10.102b}$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}. \tag{10.102c}$$

Consider a charged spherical shell of radius a consisting of a charge distribution in the polar angle alone,

$$\rho(\mathbf{r}') = \sigma(\theta')\,\delta(r'-a). \tag{10.103}$$

The electric potential *on the z-axis*, $\theta = 0$ and $\phi = 0$, is then given by

$$\begin{aligned}\phi(r,0,0) &= \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{2\pi a^2}{4\pi\varepsilon_0}\int_0^\pi \sin\theta' d\theta' \frac{\sigma(\theta')}{\sqrt{r^2+a^2-2ar\cos\theta'}},\end{aligned} \tag{10.104}$$

after evaluating the r' and ϕ' integral.

(a) Consider a uniform charge distribution on the shell,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_0(\cos\theta). \tag{10.105}$$

Evaluate the integral in Eq. (10.104) to show that

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_>}, \tag{10.106}$$

where $r_< = \text{Min}(a,r)$ and $r_> = \text{Max}(a,r)$.
Note: This was done in class. Nevertheless, present the relevant steps.

(b) Next, consider a (pure dipole, 2×1-pole,) charge distribution of the form,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_1(\cos\theta). \tag{10.107}$$

Evaluate the integral in Eq. (10.104) to show that

$$\phi(r, 0, 0) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{3} \frac{1}{r_>} \left(\frac{r_<}{r_>}\right). \tag{10.108}$$

Note: This was done in class. Nevertheless, present the relevant steps.

(c) Next, consider a (pure quadrapole, 2×2-pole,) charge distribution of the form,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_2(\cos\theta). \tag{10.109}$$

Evaluate the integral in Eq. (10.104) to show that

$$\phi(r, 0, 0) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{5} \frac{1}{r_>} \left(\frac{r_<}{r_>}\right)^2. \tag{10.110}$$

(d) For a (pure $2l$-pole) charge distribution

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_l(\cos\theta) \tag{10.111}$$

the integral in Eq. (10.104) leads to

$$\phi(r, 0, 0) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{(2l+1)} \frac{1}{r_>} \left(\frac{r_<}{r_>}\right)^l. \tag{10.112}$$

Note: No work needs to be submitted for this part. We will prove this in class.

5. (**20 points.**) Calculate the dipole moment

$$\mathbf{d} = \int d^3r\, \mathbf{r}\, \rho(\mathbf{r}) \tag{10.113}$$

of a charged spherical shell of radius a with charge density

$$\rho(\mathbf{r}) = \frac{Q}{4\pi a^2} P_1(\cos\theta)\delta(r - a). \tag{10.114}$$

6. (**20 points.**) The surface charge densities on the surface of two separate and independent charged spheres are given by

$$\sigma_1(\theta, \phi) = \frac{Q}{4\pi a^2} \cos\theta, \tag{10.115}$$

$$\sigma_2(\theta, \phi) = \frac{Q}{4\pi a^2} \cos^2\theta, \tag{10.116}$$

where θ is the polar angle in spherical coordinates. Calculate the total charge on each sphere by integrating over the surface of each sphere.

10.4 Multipole expansion

1. (**20 points.**) Consider a configuration of charges $q_1, q_2, q_3, \ldots$, at positions $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots$, and let $\mathbf{r}_0$ be an arbitrary point in space. Define the postion vector of the charges with respect to $\mathbf{r}_0$ to be

$$\mathbf{R}_i = \mathbf{r}_i - \mathbf{r}_0. \tag{10.117}$$

The monopole moment, the dipole moment, and the quadrupole moment of this configuration is given by

$$Q = q_1 + q_2 + q_3 + \ldots, \tag{10.118a}$$
$$\mathbf{d} = q_1\mathbf{R}_1 + q_2\mathbf{R}_2 + q_3\mathbf{R}_3 + \ldots, \tag{10.118b}$$
$$\mathbf{q} = q_1(3\mathbf{R}_1\mathbf{R}_1 - R_1^2\mathbf{1}) + q_2(3\mathbf{R}_2\mathbf{R}_2 - R_2^2\mathbf{1}) + q_3(3\mathbf{R}_3\mathbf{R}_3 - R_3^2\mathbf{1}) + \ldots, \tag{10.118c}$$

respectively. Evaluate the monopole moment, the dipole moment, and the quadrupole moment of three identical charges, each having charge q, positioned on the x axis at a, $2a$, and $3a$, respectively.

2. (**20 points.**) Given the quadrupole tensor

$$\mathbf{q} = q_1(3\mathbf{R}_1\mathbf{R}_1 - R_1^2\mathbf{1}) + q_2(3\mathbf{R}_2\mathbf{R}_2 - R_2^2\mathbf{1}) + q_3(3\mathbf{R}_3\mathbf{R}_3 - R_3^2\mathbf{1}) + \ldots, \tag{10.119}$$

show that

$$\operatorname{tr}\mathbf{q} = 0. \tag{10.120}$$

3. (**20 points.**) The monopole moment, the dipole moment, and the quadrupole moment, of a charge distribution $\rho(\mathbf{r})$ is given by

$$Q = \int d^3r\,\rho(\mathbf{r}), \tag{10.121a}$$
$$\mathbf{d} = \int d^3r\,\rho(\mathbf{r})\,\mathbf{r}, \tag{10.121b}$$
$$\mathbf{q} = \int d^3r\,\rho(\mathbf{r})\Big[3\mathbf{r}\mathbf{r} - r^2\mathbf{1}\Big], \tag{10.121c}$$

respectively. Consider a charge distribution consisting of a single point charge. If it is placed at the origin calculate the monopole moment, dipole moment, and quadrupole moment, of the charge distribution. Repeat the calculation if the position of the point charge is $(a, 0, 0)$.

4. (**20 points.**) Show that a configuration consisting of three charges with zero electric monopole moment and zero electric dipole moment is collinear.
Hint: Let the three charges be q_1, q_2, and q_3, and their positions be $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, respectively. Show that we can express $(\mathbf{r}_1 - \mathbf{r}_3) = a(\mathbf{r}_1 - \mathbf{r}_2)$ and $(\mathbf{r}_2 - \mathbf{r}_3) = b(\mathbf{r}_1 - \mathbf{r}_2)$. Find a and b.

5. (**20 points.**) We have three charges q_1, q_2, and q_3, at positions $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, respectively. If the configuration has zero electric monopole moment and zero electric dipole moment, then show that the three charges are collinear. Further, show that the electric quadrupole moment of the configuration is

$$\mathbf{q} = q_h\Big[3(\mathbf{r}_1 - \mathbf{r}_2)(\mathbf{r}_1 - \mathbf{r}_2) - (\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{r}_1 - \mathbf{r}_2)\mathbf{1}\Big]. \tag{10.122}$$

where q_h is the harmonic mean of q_1 and q_2 given by

$$\frac{1}{q_h} = \frac{1}{q_1} + \frac{1}{q_2}. \tag{10.123}$$

6. (**20 points.**) Two charges with charge $+q$ and $-q$ are placed at positions $\mathbf{r}_1$ and $\mathbf{r}_2$. Find the monopole moment and the dipole moment of this configuration of two charges. Is the dipole moment independent of the choice of origin? Is the dipole moment independent of the orientation of the coordinate axis?

7. (**20 points.**) Two charges with charge $+q$ each are placed at $(a, 0, 0)$ and $(-a, 0, 0)$. A third charge with charge $-2q$ is placed at the origin. Find the monopole moment, the dipole moment, and the quadrupole moment, of this configuration of the two charges.

8. (**20 points.**) Two electrons and two protons are placed at the corners of a square of length a, such that the electrons are at diagonally opposite corners. For simplicity let us choose them to be in the xy plane. Find the monopole moment, the dipole moment, and the quadrupole moment, of this configuration of four charges. Do these moments depend on the orientation of the square in the xy plane?

9. (**20 points.**) Two electrons and two protons are placed at the corners of a rectangle of length a and width b, such that the electrons are at diagonally opposite corners. For simplicity let us choose them to be in the xy plane. Find the monopole moment, the dipole moment, and the quadrupole moment, of this configuration of four charges. Do these moments depend on the orientation of the rectangle in the xy plane?

10. (**20 points.**) A positive charge q is placed at $(a, 0, 0)$. Two negative charges of charge $-q$ each are placed at $(-a/2, a\sqrt{3}/2, 0)$ and $(-a/2, -a\sqrt{3}/2, 0)$. Find the monopole moment, dipole moment, and the quadrupole moment, of this configuration of charges.

11. (**20 points.**) Two charges, each with charge $+q$, are placed at positions $\mathbf{r}_1 = a\,\hat{\mathbf{i}}$ and $\mathbf{r}_2 = a\,\hat{\mathbf{j}}$. A third charge with charge $-2q$ is placed at the origin. Find the monopole moment and the dipole moment of this configuration of three charges.

12. (**20 points.**) Two charges, each with charge $+q$, are placed at positions $\mathbf{r}_1 = a\,\hat{\mathbf{i}}$ and $\mathbf{r}_2 = a\,\hat{\mathbf{j}}$. Another set of two charges, each with charge $-q$, are placed at positions $\mathbf{r}_3 = -a\,\hat{\mathbf{i}}$ and $\mathbf{r}_4 = -a\,\hat{\mathbf{j}}$. Find the monopole moment, the dipole moment, and the quadrupole moment, of this configuration of four charges.

13. (**20 points.**) Evaluate the monopole moment, the dipole moment, and the quadrupole moment of countable infinite identical charges, each having charge q, positioned on the x axis at $a, a/2, a/3, \ldots$, respectively. Hint: Express the moments in terms of the Riemann zeta function $\zeta(s)$, which is well defined and finite for the particular values of s here.

10.5 Electric potential

1. (**40 points.**) Find the electric potential due to a uniformly charged ring of radius a and total charge Q everywhere.

   (a) Let the ring be infinitely thin. Let it be placed on the x-y plane with its center at the origin. Show that the charge density for the ring in spherical coordinates can be expressed in the form
   $$\rho(\mathbf{r}') = \frac{Q}{2\pi a} \frac{\delta\left(\theta' - \frac{\pi}{2}\right)}{r'} \delta(r' - a). \tag{10.124}$$
   Verify that $\int d^3r' \rho(\mathbf{r}') = Q$.

   (b) Using symmetry argue that the electric potential has no dependence in the azimuth angle ϕ. Thus,
   $$\phi(\mathbf{r}) = \phi(r, \theta). \tag{10.125}$$
   We will obtain a solution for the electric potential as an expansion in Legendre polynomials.

   (c) Starting from
   $$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{10.126}$$
   find the solution for the electric potential on the z axis (where $\theta = 0$) to be
   $$\phi(r, 0) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{\sqrt{a^2 + r^2}}. \tag{10.127}$$

Using the binomial expansion

$$\frac{1}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty} x^{2n} \frac{(-1)^n}{2^{2n}} \frac{(2n)!}{(n!)^2} \tag{10.128}$$

express the electric potential on the z axis in the form

$$\phi(r,0) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r_>} \sum_{n=0}^{\infty} \left(\frac{r_<}{r_>}\right)^{2n} \frac{(-1)^n}{2^{2n}} \frac{(2n)!}{(n!)^2}, \tag{10.129}$$

where $r_< = \text{Min}(r,a)$ and $r_> = \text{Max}(r,a)$.

(d) Let the Legendre expansion of the electric potential be

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{a} \sum_{l=0}^{\infty} A_l(r) P_l(\cos\theta). \tag{10.130}$$

The electric potential satisfies the Laplacian

$$-\nabla^2\phi = 0 \tag{10.131}$$

for points not on the ring. Using the Laplacian in spherical coordinates and the differential equation satisfied by the Legendre polynomials, deduce the differential equation for the coefficients $A_l(r)$ to be

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] A_l(r) = 0. \tag{10.132}$$

Show that

$$A_l(r) = \alpha_l \left(\frac{r}{a}\right)^l + \beta_l \left(\frac{a}{r}\right)^{l+1}. \tag{10.133}$$

Thus, the Legendre expansion for the electric potential is

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{a} \sum_{l=0}^{\infty} \left[\alpha_l \left(\frac{r}{a}\right)^l + \beta_l \left(\frac{a}{r}\right)^{l+1}\right] P_l(\cos\theta). \tag{10.134}$$

Requiring the boundary condition that the electric potential be zero for $r \to \infty$ and is finite at $r = 0$, show that

$$\phi(r,\theta) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0} \dfrac{Q}{a} \displaystyle\sum_{l=0}^{\infty} \alpha_l \left(\frac{r}{a}\right)^l P_l(\cos\theta), & r < a, \\ \dfrac{1}{4\pi\varepsilon_0} \dfrac{Q}{r} \displaystyle\sum_{l=0}^{\infty} \beta_l \left(\frac{a}{r}\right)^l P_l(\cos\theta), & a < r. \end{cases} \tag{10.135}$$

(e) Using Eq. (10.135), we have

$$\phi(r,0) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0} \dfrac{Q}{a} \displaystyle\sum_{l=0}^{\infty} \alpha_l \left(\frac{r}{a}\right)^l, & r < a, \\ \dfrac{1}{4\pi\varepsilon_0} \dfrac{Q}{r} \displaystyle\sum_{l=0}^{\infty} \beta_l \left(\frac{a}{r}\right)^l, & a < r. \end{cases} \tag{10.136}$$

where we used $P_l(1) = 1$. Comparing Eqs. (10.129) and (10.136) show that

$$\alpha_l = \beta_l = \begin{cases} 0 & l = 1,3,5,\ldots, \\ \dfrac{(-1)^n}{2^{2n}} \dfrac{(2n)!}{(n!)^2}, & l = 2n, \quad n = 0,1,2,\ldots. \end{cases} \tag{10.137}$$

Thus, show that

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r_>} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{(2n)!}{(n!)^2} \left(\frac{r_<}{r_>}\right)^{2n} P_{2n}(\cos\theta). \tag{10.138}$$

2. (**40 points.**) Let us consider a uniformly charged circular disc of radius a and total charge Q. Let the disc be infinitely thin. Let it be placed on the x-y plane with its center at the origin.

   (a) Show that the charge density for the disc in spherical coordinates can be expressed in the form

$$\rho(\mathbf{r}') = \frac{Q}{\pi a^2} \frac{\delta\left(\theta' - \frac{\pi}{2}\right)}{r'} \theta(a - r'). \tag{10.139}$$

   Verify that $\int d^3r' \rho(\mathbf{r}') = Q$.

   (b) Using symmetry argue that the electric potential has no dependence in the azimuth angle ϕ. Thus,

$$\phi(\mathbf{r}) = \phi(r, \theta). \tag{10.140}$$

   Our goal here will be to obtain a solution for the electric potential as an expansion in Legendre polynomials.

   (c) Starting from

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{10.141}$$

   find the solution for the electric potential on the z axis (where $\theta = 0$) to be

$$\phi(r, 0) = \frac{1}{4\pi\varepsilon_0} \frac{2Q}{a^2} \left[\sqrt{a^2 + r^2} - r\right]. \tag{10.142}$$

   Using the binomial expansion

$$\sqrt{1 + x^2} = 1 + \sum_{n=1}^{\infty} x^{2n} \frac{(-1)^{n-1}}{2n} \frac{[2(n-1)]!}{2^{2(n-1)}[(n-1)!]^2} \tag{10.143}$$

   express the electric potential on the z axis in the form

$$\phi(r, 0) = \begin{cases} \frac{1}{4\pi\varepsilon_0} \frac{2Q}{a} \left[1 - \frac{r}{a} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n} \frac{[2(n-1)]!}{2^{2(n-1)}[(n-1)!]^2} \left(\frac{r}{a}\right)^{2n}\right], & r < a, \\ \frac{1}{4\pi\varepsilon_0} \frac{2Q}{r} \sum_{n=0}^{\infty} \frac{(-1)^n}{2(n+1)} \frac{(2n)!}{2^{2n}(n!)^2} \left(\frac{a}{r}\right)^{2n}, & a < r. \end{cases} \tag{10.144}$$

   (d) Let the Legendre expansion of the electric potential be

$$\phi(r, \theta) = \frac{1}{4\pi\varepsilon_0} \frac{2Q}{a} \sum_{l=0}^{\infty} A_l(r) P_l(\cos\theta). \tag{10.145}$$

   The electric potential satisfies the Laplacian

$$-\nabla^2 \phi = 0 \tag{10.146}$$

   outside the disc. Using the Laplacian in spherical coordinates and the differential equation satisfied by the Legendre polynomials, deduce the differential equation for the coefficients $A_l(r)$ to be

$$\left[\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] A_l(r) = 0. \tag{10.147}$$

   Show that

$$A_l(r) = \alpha_l \left(\frac{r}{a}\right)^l + \beta_l \left(\frac{a}{r}\right)^{l+1}. \tag{10.148}$$

Thus, the Legendre expansion for the electric potential is

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{a}\sum_{l=0}^{\infty}\left[\alpha_l\left(\frac{r}{a}\right)^l + \beta_l\left(\frac{a}{r}\right)^{l+1}\right]P_l(\cos\theta). \tag{10.149}$$

Requiring the boundary condition that the electric potential be zero for $r \to \infty$ and is finite at $r = 0$, show that

$$\phi(r,\theta) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{a}\displaystyle\sum_{l=0}^{\infty}\alpha_l\left(\frac{r}{a}\right)^l P_l(\cos\theta), & r < a, \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{r}\displaystyle\sum_{l=0}^{\infty}\beta_l\left(\frac{a}{r}\right)^l P_l(\cos\theta), & a < r. \end{cases} \tag{10.150}$$

(e) Using Eq. (10.150), we have

$$\phi(r,0) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{a}\displaystyle\sum_{l=0}^{\infty}\alpha_l\left(\frac{r}{a}\right)^l, & r < a, \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{r}\displaystyle\sum_{l=0}^{\infty}\beta_l\left(\frac{a}{r}\right)^l, & a < r. \end{cases} \tag{10.151}$$

where we used $P_l(1) = 1$. Comparing Eqs. (10.144) and (10.151) show that

$$\alpha_l = \begin{cases} 1 & l = 0, \\ -1 & l = 1, \\ 0 & l = 3, 5, 7, \ldots, \\ \dfrac{(-1)^{n-1}}{2n}\dfrac{[2(n-1)]!}{2^{2(n-1)}[(n-1)!]^2}, & l = 2n, \quad n = 1, 2, 3, \ldots, \end{cases} \tag{10.152}$$

and

$$\beta_l = \begin{cases} 0 & l = 1, 3, 5, \ldots, \\ \dfrac{(-1)^n}{2(n+1)}\dfrac{(2n)!}{2^{2n}(n!)^2} & l = 2n, \quad n = 0, 1, 2, 3, \ldots. \end{cases} \tag{10.153}$$

Thus, show that

$$\phi(r,\theta) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{a}\left[1 - \dfrac{r}{a}P_1(\cos\theta) + \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n}\frac{[2(n-1)]!}{2^{2(n-1)}[(n-1)!]^2}\left(\frac{r}{a}\right)^{2n}P_{2n}(\cos\theta)\right], & r < a, \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{r}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2(n+1)}\frac{(2n)!}{2^{2n}(n!)^2}\left(\frac{a}{r}\right)^{2n}P_{2n}(\cos\theta), & a < r. \end{cases} \tag{10.154}$$

(f) For $r \ll a$ the disc should simulate a plate of infinite extent. Show that

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{a}\left[1 - \frac{z}{a}\right] + \mathcal{O}\left(\frac{z}{a}\right)^2, \tag{10.155}$$

using $rP_1(\cos\theta) = z$. This leads to the electric field for a plate of infinite extent,

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi = \hat{\mathbf{z}}\frac{\sigma}{2\varepsilon_0}, \tag{10.156}$$

where $\sigma = Q/(\pi a^2)$.