

Homework No. 09 (2026 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2026 Mar 24, 4.30pm

1. (20 points.) Consider infinitesimal rigid translation in space, described by

$$\delta\mathbf{r} = \delta\boldsymbol{\epsilon}, \quad \delta\mathbf{p} = 0, \quad \delta t = 0, \quad (1)$$

where $\delta\boldsymbol{\epsilon}$ is independent of position and time.

- (a) Show that the change in the action due to the above translation is

$$\frac{\delta W}{\delta\boldsymbol{\epsilon}} = - \int_{t_1}^{t_2} dt \frac{\partial H}{\partial \mathbf{r}}. \quad (2)$$

- (b) Show, separately, that the change in the action under the above translation is also given by

$$\frac{\delta W}{\delta\boldsymbol{\epsilon}} = \int_{t_1}^{t_2} dt \frac{d\mathbf{p}}{dt} = \mathbf{p}(t_2) - \mathbf{p}(t_1). \quad (3)$$

- (c) The system is defined to have translational symmetry when the action does not change under rigid translation. Show that a system has translation symmetry when

$$-\frac{\partial H}{\partial \mathbf{r}} = 0. \quad (4)$$

That is, when the Hamiltonian is independent of position. Or, when the force $\mathbf{F} = -\partial H/\partial \mathbf{r} = 0$.

- (d) Deduce that the linear momentum is conserved, that is,

$$\mathbf{p}(t_1) = \mathbf{p}(t_2), \quad (5)$$

when the action has translation symmetry.

2. (20 points.) Consider infinitesimal rigid translation in time, described by

$$\delta\mathbf{r} = 0, \quad \delta\mathbf{p} = 0, \quad \delta t = \delta\epsilon, \quad (6)$$

where $\delta\epsilon$ is independent of position and time.

- (a) Show that the change in the action due to the above translation is

$$\frac{\delta W}{\delta\epsilon} = - \int_{t_1}^{t_2} dt \frac{\partial H}{\partial t}. \quad (7)$$

- (b) Show, separately, that the change in the action under the above translation is also given by

$$\frac{\delta W}{\delta\epsilon} = - \int_{t_1}^{t_2} dt \frac{dH}{dt} = -H(t_2) + H(t_1). \quad (8)$$

- (c) The system is defined to have translational symmetry when the action does not change under rigid translation. Show that a system has translation symmetry when

$$-\frac{\partial H}{\partial t} = 0. \quad (9)$$

That is, when the Hamiltonian is independent of time.

- (d) Deduce that the Hamiltonian is conserved, that is,

$$H(t_1) = H(t_2), \quad (10)$$

when the action has translation symmetry.

3. (20 points.) A general rotation in 3-dimensions can be written in terms of consecutive rotations about x , y , and z axes,

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & \sin \theta_1 \\ 0 & -\sin \theta_1 & \cos \theta_1 \end{pmatrix} \begin{pmatrix} \cos \theta_2 & 0 & -\sin \theta_2 \\ 0 & 1 & 0 \\ \sin \theta_2 & 0 & \cos \theta_2 \end{pmatrix} \begin{pmatrix} \cos \theta_3 & \sin \theta_3 & 0 \\ -\sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \quad (11)$$

For infinitesimal rotations we use

$$\cos \theta_i \sim 1, \quad (12a)$$

$$\sin \theta_i \sim \theta_i \rightarrow \delta \theta_i, \quad (12b)$$

to obtain

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} 1 & \delta \theta_3 & -\delta \theta_2 \\ -\delta \theta_3 & 1 & \delta \theta_1 \\ \delta \theta_2 & -\delta \theta_1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \quad (13)$$

Show that this corresponds to the vector relation

$$\mathbf{r}' = \mathbf{r} - \delta \boldsymbol{\theta} \times \mathbf{r} \quad (14)$$

such that

$$\delta \mathbf{r} = \delta \boldsymbol{\theta} \times \mathbf{r}, \quad (15)$$

where

$$\mathbf{r} = x_1 \hat{\mathbf{x}} + x_2 \hat{\mathbf{y}} + x_3 \hat{\mathbf{z}}, \quad (16a)$$

$$\delta \boldsymbol{\theta} = \delta \theta_1 \hat{\mathbf{x}} + \delta \theta_2 \hat{\mathbf{y}} + \delta \theta_3 \hat{\mathbf{z}}. \quad (16b)$$

As a particular example, verify that a rotation about the direction $\hat{\mathbf{z}}$ by an infinitesimal (azimuth) angle $\delta \phi$ is described by

$$\delta \boldsymbol{\theta} = \hat{\mathbf{z}} \delta \phi. \quad (17)$$

The corresponding infinitesimal transformation in \mathbf{r} is given by

$$\delta \mathbf{r} = \delta \phi \hat{\mathbf{z}} \times \mathbf{r} = \hat{\boldsymbol{\phi}} \rho \delta \phi, \quad (18)$$

where ρ and ϕ are the cylindrical coordinates defined as

$$\hat{\mathbf{z}} \times \mathbf{r} = \hat{\boldsymbol{\phi}} \quad \text{and} \quad |\hat{\mathbf{z}} \times \mathbf{r}| = \rho. \quad (19)$$

Observe that, in rectangular coordinates $\rho \hat{\boldsymbol{\phi}} = x \hat{\mathbf{y}} - y \hat{\mathbf{x}}$.

4. (20 points.) Consider infinitesimal rigid rotation, described by

$$\delta \mathbf{r} = \delta \boldsymbol{\theta} \times \mathbf{r}, \quad \delta \mathbf{p} = \delta \boldsymbol{\theta} \times \mathbf{p}, \quad \delta t = 0, \quad (20)$$

where $d\delta \boldsymbol{\theta}/dt = 0$.

(a) Show that the variation in the action under the above rotation is

$$\frac{\delta W}{\delta \theta} = \int_{t_1}^{t_2} dt \left[\mathbf{r} \times \frac{\partial L}{\partial \mathbf{r}} + \mathbf{p} \times \frac{\partial L}{\partial \mathbf{p}} \right] \quad (21)$$

or

$$\frac{\delta W}{\delta \theta} = - \int_{t_1}^{t_2} dt \left[\mathbf{r} \times \frac{\partial H}{\partial \mathbf{r}} + \mathbf{p} \times \frac{\partial H}{\partial \mathbf{p}} \right]. \quad (22)$$

(b) Show, separately, that the change in the action under the above rotation is also given by

$$\frac{\delta W}{\delta \theta} = \int_{t_1}^{t_2} dt \frac{d\mathbf{L}}{dt} = \mathbf{L}(t_2) - \mathbf{L}(t_1), \quad (23)$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum.

(c) The system is defined to have rotational symmetry when the action does not change under rigid rotation. Show that a system has rotation symmetry when

$$\mathbf{r} \times \frac{\partial L}{\partial \mathbf{r}} = 0 \quad \text{and} \quad \mathbf{p} \times \frac{\partial L}{\partial \mathbf{p}} = 0, \quad (24)$$

or

$$\mathbf{r} \times \frac{\partial H}{\partial \mathbf{r}} = 0 \quad \text{and} \quad \mathbf{p} \times \frac{\partial H}{\partial \mathbf{p}} = 0. \quad (25)$$

Show that this corresponds to

$$\frac{\partial L}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \phi} = 0, \quad (26)$$

or

$$\frac{\partial H}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial H}{\partial \phi} = 0. \quad (27)$$

That is, when the Lagrangian is independent of angular coordinates θ and ϕ .

(d) Deduce that the angular momentum is conserved, that is,

$$\mathbf{L}(t_1) = \mathbf{L}(t_2), \quad (28)$$

when the action has rotational symmetry.