

Homework No. 09 (2026 Spring)

PHYS 510: CLASSICAL MECHANICS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Tuesday, 2026 Apr 7, 4.30pm

1. (20 points.) For two functions

$$A = A(\mathbf{x}, \mathbf{p}, t), \quad (1a)$$

$$B = B(\mathbf{x}, \mathbf{p}, t), \quad (1b)$$

the Poisson bracket with respect to the canonical variables \mathbf{x} and \mathbf{p} is defined as

$$[A, B]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} \equiv \frac{\partial A}{\partial \mathbf{x}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{x}}. \quad (2)$$

Show that the Poisson bracket satisfies the conditions for a Lie algebra. That is, show that

- (a) Antisymmetry:

$$[A, B]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} = -[B, A]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}}. \quad (3)$$

- (b) Bilinearity: (a and b are numbers.)

$$[aA + bB, C]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} = a[A, C]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} + b[B, C]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}}. \quad (4)$$

Further show that

$$[AB, C]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} = A[B, C]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} + [A, C]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} B. \quad (5)$$

- (c) Jacobi's identity:

$$[A, [B, C]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}}]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} + [B, [C, A]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}}]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} + [C, [A, B]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}}]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} = 0. \quad (6)$$

2. (20 points.) Show that the commutator of two matrices,

$$[\mathbf{A}, \mathbf{B}] \equiv \mathbf{AB} - \mathbf{BA}, \quad (7)$$

satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that

- (a) Antisymmetry:

$$[\mathbf{A}, \mathbf{B}] = -[\mathbf{B}, \mathbf{A}]. \quad (8)$$

- (b) Bilinearity: (a and b are numbers.)

$$[a\mathbf{A} + b\mathbf{B}, \mathbf{C}] = a[\mathbf{A}, \mathbf{C}] + b[\mathbf{B}, \mathbf{C}]. \quad (9)$$

Further show that

$$[\mathbf{AB}, \mathbf{C}] = \mathbf{A}[\mathbf{B}, \mathbf{C}] + [\mathbf{A}, \mathbf{C}]\mathbf{B}. \quad (10)$$

- (c) Jacobi's identity:

$$[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] + [\mathbf{B}, [\mathbf{C}, \mathbf{A}]] + [\mathbf{C}, [\mathbf{A}, \mathbf{B}]] = 0. \quad (11)$$

3. (20 points.) Show that the vector product of two vectors, in this problem denoted using

$$[\mathbf{A}, \mathbf{B}]_v \equiv \mathbf{A} \times \mathbf{B}, \quad (12)$$

satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that

(a) Antisymmetry:

$$[\mathbf{A}, \mathbf{B}]_v = -[\mathbf{B}, \mathbf{A}]_v. \quad (13)$$

(b) Bilinearity: (a and b are numbers.)

$$[a\mathbf{A} + b\mathbf{B}, \mathbf{C}]_v = a[\mathbf{A}, \mathbf{C}]_v + b[\mathbf{B}, \mathbf{C}]_v. \quad (14)$$

Further show that

$$[\mathbf{A} \times \mathbf{B}, \mathbf{C}]_v = \mathbf{A} \times [\mathbf{B}, \mathbf{C}]_v + [\mathbf{A}, \mathbf{C}]_v \times \mathbf{B}. \quad (15)$$

(c) Jacobi's identity:

$$[\mathbf{A}, [\mathbf{B}, \mathbf{C}]_v]_v + [\mathbf{B}, [\mathbf{C}, \mathbf{A}]_v]_v + [\mathbf{C}, [\mathbf{A}, \mathbf{B}]_v]_v = 0. \quad (16)$$

4. (20 points.) (Refer Sec. 21 Dirac's QM book.)

The product rule for Poisson bracket can be stated in the following different forms:

$$[A_1 A_2, B]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} = A_1 [A_2, B]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} + [A_1, B]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} A_2, \quad (17a)$$

$$[A, B_1 B_2]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} = B_1 [A, B_2]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} + [A, B_1]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} B_2. \quad (17b)$$

(a) Thus, evaluate, in two different ways,

$$\begin{aligned} [A_1 A_2, B_1 B_2]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} &= A_1 B_1 [A_2, B_2]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} + A_1 [A_2, B_1]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} B_2 \\ &\quad + B_1 [A_1, B_2]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} A_2 + [A_1, B_1]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} B_2 A_2, \end{aligned} \quad (18a)$$

$$\begin{aligned} [A_1 A_2, B_1 B_2]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} &= B_1 A_1 [A_2, B_2]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} + B_1 [A_1, B_2]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} A_2 \\ &\quad + A_1 [A_2, B_1]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} B_2 + [A_1, B_1]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} A_2 B_2. \end{aligned} \quad (18b)$$

(b) Subtracting these results, obtain

$$(A_1 B_1 - B_1 A_1) [A_2, B_2]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} = [A_1, B_1]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} (A_2 B_2 - B_2 A_2). \quad (19)$$

Thus, using the definition of the commutation relation,

$$[A, B] \equiv AB - BA, \quad (20)$$

obtain the relation

$$[A_1, B_1] [A_2, B_2]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} = [A_1, B_1]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} [A_2, B_2]. \quad (21)$$

(c) Since this condition holds for A_1 and B_1 independent of A_2 and B_2 , conclude that

$$[A_1, B_1] = i\hbar [A_1, B_1]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}}, \quad (22a)$$

$$[A_2, B_2] = i\hbar [A_2, B_2]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}}, \quad (22b)$$

where $i\hbar$ is necessarily a constant, independent of A_1 , A_2 , B_1 , and B_2 . This is the connection between the commutator bracket in quantum mechanics and the Poisson bracket in classical mechanics. If A 's and B 's are numbers, then, because their commutation relation is equal to zero, we necessarily have $\hbar = 0$. But, if the commutation relation of A 's and B 's is not zero, then finite values of \hbar is allowed.

(d) Here the imaginary number $i = \sqrt{-1}$. Show that the constant \hbar is a real number if we presume the Poisson bracket to be real, and require the construction

$$C = \frac{1}{i}(AB - BA) \quad (23)$$

to be Hermitian. Experiment dictates that $\hbar = h/2\pi$, where

$$h \sim 6.63 \times 10^{-34} \text{ J}\cdot\text{s} \quad (24)$$

is Planck's constant with dimensions of action.

5. (20 points.) Hamiltonian for the motion of a ball (along the radial direction) near the surface of Earth is given by

$$H(z, p_z) = \frac{p_z^2}{2m} - mgz. \quad (25)$$

- (a) Determine the equations of motions using

$$\frac{dz}{dt} = \frac{\partial H}{\partial p_z} \quad \text{and} \quad \frac{dp_z}{dt} = -\frac{\partial H}{\partial z}. \quad (26)$$

Then, solve the coupled differential equations to find the familiar elementary solution

$$z = z_0 + \frac{p_0}{m}t + \frac{1}{2}gt^2 \quad (27a)$$

and

$$p_z = p_{z,0} + mgt. \quad (27b)$$

- (b) Next, determine the equations of motion using

$$\frac{dz}{dt} = [z, H]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} \quad \frac{dp_z}{dt} = [p_z, H]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}}. \quad (28)$$

Evaluate

$$\frac{d^2z}{dt^2} = [[z, H]_{\mathbf{x}, \mathbf{p}, 0}^{\text{P.B.}}, H]_{\mathbf{x}, \mathbf{p}, 0}^{\text{P.B.}} \quad (29a)$$

$$\frac{d^3z}{dt^3} = [[[z, H]_{\mathbf{x}, \mathbf{p}, 0}^{\text{P.B.}}, H]_{\mathbf{x}, \mathbf{p}, 0}^{\text{P.B.}}, H]_{\mathbf{x}, \mathbf{p}, 0}^{\text{P.B.}} \quad (29b)$$

⋮

and

$$\frac{d^2p_z}{dt^2} = [[p_z, H]_{\mathbf{x}, \mathbf{p}, 0}^{\text{P.B.}}, H]_{\mathbf{x}, \mathbf{p}, 0}^{\text{P.B.}} \quad (30a)$$

$$\frac{d^3p_z}{dt^3} = [[[p_z, H]_{\mathbf{x}, \mathbf{p}, 0}^{\text{P.B.}}, H]_{\mathbf{x}, \mathbf{p}, 0}^{\text{P.B.}}, H]_{\mathbf{x}, \mathbf{p}, 0}^{\text{P.B.}} \quad (30b)$$

⋮

Then, using

$$z = z_0 + t \left\{ \frac{dz}{dt} \right\}_0 + \frac{1}{2}t^2 \left\{ \frac{d^2z}{dt^2} \right\}_0 + \dots \quad (31)$$

and

$$p_z = p_{z,0} + t \left\{ \frac{dp_z}{dt} \right\}_0 + \frac{1}{2}t^2 \left\{ \frac{d^2p_z}{dt^2} \right\}_0 + \dots \quad (32)$$

rederive the solutions in Eqs. (27). Here 0 in the subscripts refers to the initial conditions at $t = 0$.

6. (20 points.) Harmonic oscillations are described by the Hamiltonian

$$H(x, p) = \frac{1}{2}p^2 + \frac{1}{2}x^2. \quad (33)$$

(a) Determine the equations of motions using

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \quad \text{and} \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x}. \quad (34)$$

Then, solve the coupled differential equations to find the solutions

$$x = x_0 \cos t + p_0 \sin t, \quad (35a)$$

$$p = -x_0 \sin t + p_0 \cos t, \quad (35b)$$

where x_0 and p_0 are given using the initial conditions at $t = 0$.

(b) Next, determine the equations of motion using

$$\frac{dx}{dt} = [x, H]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} \quad \frac{dp}{dt} = [p, H]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}}. \quad (36)$$

Evaluate

$$[\dots [[x, H]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}}, H]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}}, \dots]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} \quad (37)$$

and

$$[\dots [[p, H]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}}, H]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}}, \dots]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} \quad (38)$$

iteratively. Thus, evaluate

$$\frac{d^n x}{dt^n} = \begin{cases} x i^n, & \text{for } n = 0, 2, 4, \dots, \\ p i^{n-1}, & \text{for } n = 1, 3, 5, \dots, \end{cases} \quad (39)$$

and

$$\frac{d^n p}{dt^n} = \begin{cases} p i^n, & \text{for } n = 0, 2, 4, \dots, \\ -x i^{n-1}, & \text{for } n = 1, 3, 5, \dots \end{cases} \quad (40)$$

Then, using

$$x = x_0 + t \left\{ \frac{dx}{dt} \right\}_0 + \frac{1}{2} t^2 \left\{ \frac{d^2 x}{dt^2} \right\}_0 + \dots \quad (41)$$

and

$$p = p_0 + t \left\{ \frac{dp}{dt} \right\}_0 + \frac{1}{2} t^2 \left\{ \frac{d^2 p}{dt^2} \right\}_0 + \dots \quad (42)$$

rederive the solutions in Eqs. (35). Here 0 in the subscripts refers to the initial conditions at $t = 0$.

7. (20 points.) Given F and G are constants of motion, that is

$$[F, H]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} = 0 \quad \text{and} \quad [G, H]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}} = 0. \quad (43)$$

Then, using Jacobi's identity, show that $[F, G]_{\mathbf{x}, \mathbf{p}}^{\text{P.B.}}$ is also a constant of motion. Thus, conclude the following:

(a) If L_x and L_y are constants of motion, then L_z is also a constant of motion.

(b) If p_x and L_z are constants of motion, then p_y is also a constant of motion.