Notes on Classical Mechanics

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1. These are notes prepared for the benefit of students enrolled in Classical Mechanics (PHYS-510) at Southern Illinois University–Carbondale. It will be updated periodically, and will evolve during the semester. It is not a substitute for standard textbooks, but a supplement prepared as a study-guide.

2. The following textbooks were extensively used in this compilation.

   (a) Calculus of Variations,
   I. M. Gelfand and S. V. Fomin, (translated by Richard A. Silverman,)
   Prentice-Hall

   (b) Mechanics: Lectures on Theoretical Physics, Volume I,
   Arnold Sommerfeld, (translated by Martin O. Stern,)
   Academic Press

   (c) Mechanics, 3rd edition, Course of Theoretical Physics, Volume 1,
   L. D. Landau and E. M. Lifshitz, (translated by J. B. Sykes and J. S. Bell,)
   Elsevier

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Contents

Chapter 1

Newton's laws of motion

1.1 Position dependent forces

1. (**20 points.**) Radial free fall of a meteoroid. Refer 20210121 video.
2. (**20 points.**) (Refer Landau and Lifshitz, Problem 1 in Chapter 3.) A simple pendulum consists of a particle of mass m suspended by a massless rod of length l in a uniform gravitational field g.
   (a) Identify the two forces acting on the pendulum to be the force of gravity $m\mathbf{g}$ and the force of tension $\mathbf{T}$. Thus, deduce the Newton equation of motion to be
   $$m\mathbf{a} = m\mathbf{g} + \mathbf{T}, \tag{1.1}$$
   where $\mathbf{a}$ is acceleration of mass m. Starting from Eq. (1.1) derive the equation of motion for the simple pendulum
   $$\frac{d^2\phi}{dt^2} = -\omega_0^2 \sin\phi, \tag{1.2}$$
   where
   $$\omega_0 = \frac{2\pi}{T_0} = \sqrt{\frac{g}{l}}. \tag{1.3}$$
   (b) Starting from Eq. (1.2) derive the statement of conservation of energy for this system,
   $$\frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi = \text{constant}. \tag{1.4}$$
   Hint: Multiply Eq. (1.2) by $\dot{\phi}$ and express the equation as a total derivative with respect to time.
   (c) For initial conditions $\phi(0) = \phi_0$ and $\dot{\phi}(0) = 0$ show that
   $$\frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi = -mgl\cos\phi_0. \tag{1.5}$$
   Thus, derive
   $$\frac{dt}{T_0} = \frac{1}{2\pi}\frac{d\phi}{\sqrt{2(\cos\phi - \cos\phi_0)}} \tag{1.6}$$
   where $T_0 = 2\pi\sqrt{l/g}$.
   (d) The time period of oscillations of the simple pendulum is equal to four times the time taken between $\phi = 0$ and $\phi = \phi_0$. Thus, show that
   $$T = 4\frac{T_0}{2\pi}\int_0^{\phi_0} \frac{d\phi}{\sqrt{2(\cos\phi - \cos\phi_0)}} \tag{1.7}$$
   $$= \frac{T_0}{\pi}\int_0^{\phi_0} \frac{d\phi}{\sqrt{\sin^2\frac{\phi_0}{2} - \sin^2\frac{\phi}{2}}}. \tag{1.8}$$

Then, substitute $\sin\theta = \sin(\phi/2)/\sin(\phi_0/2)$ to determine the period of oscillations of the simple pendulum as a function of the amplitude of oscillations ϕ_0 to be

$$T = T_0 \frac{2}{\pi} K\left(\sin\frac{\phi_0}{2}\right), \tag{1.9}$$

where

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{1.10}$$

is the complete elliptic integral of the first kind.

(e) Using the power series expansion

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} \tag{1.11}$$

show that for small oscillations ($\phi_0/2 \ll 1$)

$$T = T_0\left[1 + \frac{\phi_0^2}{16} + \ldots\right]. \tag{1.12}$$

(f) Estimate the percentage error made in the approximation $T \sim T_0$ for $\phi_0 \sim 60°$.

(g) Plot the time period T of Eq. (1.9) as a function of ϕ_0. What can you conclude about the time period for $\phi_0 = \pi$?

3. (**20 points.**) Assume Earth to be a solid spherical ball of uniform density. Neglect the influence of all other stars and galaxies. Consider a hypothetical tunnel passing through the center of Earth and connecting two diameterically opposite points on the surface of Earth by a straight line. Ignore friction and the rotational motion of Earth. Use the mass of Earth to be 6.0×10^{24} kg, radius of Earth to be 6.4×10^6 m. Newton's gravitational constant is $6.7 \times 10^{-11}\,\text{Nm}^2/\text{kg}^2$.

(a) Show that the gravitational field is

$$\mathbf{g}(\mathbf{r}) = \begin{cases} -\hat{\mathbf{r}}\dfrac{GM}{R^2}\dfrac{r}{R}, & \text{for} \quad r < R, \\[2ex] -\hat{\mathbf{r}}\dfrac{GM}{r^2}, & \text{for} \quad R < r, \end{cases} \tag{1.13}$$

where G is Newton's gravitational constant and $\hat{\mathbf{r}}$ are unit vectors radiating out from the center of Earth. Plot the magnitude of $\mathbf{g}(\mathbf{r})$ as a function of r. Evaluate the gravitational field on the surface of Earth,

$$g = \frac{GM}{R^2}, \tag{1.14}$$

to significant digits, presuming that the gravitational field is continuous at the surface. Note that we could model (undetectable) exotic matter with usual mass to exist only close to the surface of Earth using a δ-function field, which we shall not attempt here.

(b) The gravitational potential associated with the gravitational field is given by the differential statement

$$\mathbf{g}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}), \tag{1.15}$$

or the integral statement

$$-d\phi = \mathbf{g}\cdot d\mathbf{r}. \tag{1.16}$$

Thus, determine the gravitational potential

$$\phi(\mathbf{r}) = \begin{cases} \dfrac{1}{2}\dfrac{GM}{R}\dfrac{r^2}{R^2} + c_1, & \text{for} \quad r < R, \\ -\dfrac{GM}{r} + c_2, & \text{for} \quad R < r. \end{cases} \tag{1.17}$$

Determine the arbitrary integration constant c_2 by choosing the gravitational potential to be zero at $r \to \infty$. Requiring the gravitational potential to be continuous at the surface show that

$$c_1 = -\frac{3}{2}\frac{GM}{R}. \tag{1.18}$$

Plot the gravitational potential with respect to r using a graphing software. Ponder if there is flexibility here while investigating exotic matter, which we shall not attempt here.

(c) Consider the free fall from a great distance, say infinity, of an object starting from rest. Determine it's velocity when it reaches the surface of Earth. This is the escape velocity of Earth,

$$v_e = \sqrt{2gR}. \tag{1.19}$$

(d) Consider a free fall starting from rest at the surface of Earth, in a frictionless tunnel passing through the center of Earth. Find it's velocity as it crosses the center of Earth to be

$$v = \frac{v_e}{\sqrt{2}}. \tag{1.20}$$

(e) Consider a free fall starting from rest at infinitely large distance falling along a radial line aligned with the tunnel. Find it's velocity as it crosses the center of Earth to be

$$v = v_e\sqrt{\frac{3}{2}}. \tag{1.21}$$

(f) Is the time taken for these free falls and escape scenarios finite? In particular, try to evaluate the time taken to escape the gravitational field of Earth.

4. (**20 points.**) Assume Earth to be a solid spherical ball of uniform density. Consider a hypothetical tunnel passing through the center of Earth and connecting two points on the surface of Earth by a straight line. Determine the time taken, (in minutes) to two siginificant digits, starting from rest, to travel from one point to the other, when a mass is dropped at one end of the tunnel. Ignore friction and the rotational motion of Earth. Use the mass of Earth to be $6.0 \times 10^{24}\,\text{kg}$, radius of Earth to be $6.4 \times 10^6\,\text{m}$. Newton's gravitational constant is $6.67 \times 10^{-11}\,\text{Nm}^2/\text{kg}^2$.

A more realistic density profile of Earth is

$$\rho(r) = \begin{cases} \rho_0, & \text{for} \quad r < \frac{R}{2}, \\ \frac{1}{2}\rho_0, & \text{for} \quad \frac{R}{2} < r < R, \end{cases} \tag{1.22}$$

where

$$\rho_0 = \frac{16}{9}\frac{M}{\frac{4\pi}{3}R^3}, \tag{1.23}$$

where R is the radius of Earth and M is the mass of Earth. Show that the above density profile leads to the following profile for the gravitational field for Earth,

$$g(r) = \begin{cases} -\dfrac{16}{9}\dfrac{GMr}{R^3}, & \text{for} \quad r < \dfrac{R}{2}, \\ -\dfrac{8}{9}\dfrac{GM}{R^2}\dfrac{1}{2}\left[\dfrac{2r}{R} + \left(\dfrac{R}{2r}\right)^2\right], & \text{for} \quad \dfrac{R}{2} < r < R, \\ -\dfrac{GM}{r^2}, & \text{for} \quad R < r, \end{cases} \tag{1.24}$$

where G is Newton's gravitational constant. Plot $g(r)$ as a function of r. Approximate the above gravitational field as

$$g(r) \approx \begin{cases} -\dfrac{GM}{R^2}\dfrac{2r}{R}, & \text{for} \quad r < \dfrac{R}{2}, \\ -\dfrac{GM}{R^2}, & \text{for} \quad \dfrac{R}{2} < r < R, \\ -\dfrac{GM}{r^2}, & \text{for} \quad R < r. \end{cases} \tag{1.25}$$

Plot the approximate gravitational field and compare it with the exact version. Argue that it is accurate to about ten percent. Determine the new time taken, (in minutes) to two siginificant digits, starting from rest, to travel from one point to the other, when a mass is dropped at one end of the tunnel. Ignore friction and the rotational motion of Earth.
Refer: The gravity tunnel in a non-uniform Earth, by Alexander R. Klotz, Am. J. Phys. 83 (2015) 231; arXiv:1308.1342.

1.2 Velocity dependent forces

1. (**20 points.**) The force for linear drag is

$$\mathbf{F} = -b\mathbf{v}. \tag{1.26}$$

For a mass m falling under uniform gravity in the presence of linear drag, with velocity chosen to be positive for upward direction, we have the equation of motion

$$m\frac{dv}{dt} = -mg - bv. \tag{1.27}$$

For the case when the mass is moving vertically down, as the mass falls it gains speed and the drag force eventually balances the force of gravity, and from this point on it does not accelerate. Thus, the terminal velocity is defined by requiring $dv/dt = 0$, that is,

$$v_T = \frac{mg}{b} \tag{1.28}$$

downwards. The equation of motion can be solved for the initial condition of the particle starting from rest, $v(0) = 0$, which leads to the solution

$$v(t) = v_T\Big(e^{-\frac{t}{\tau}} - 1\Big), \tag{1.29}$$

where

$$\tau = \frac{v_T}{g} \tag{1.30}$$

sets the scale for time in the problem and is illustrated in Figure 1.1.

(a) Evaluate the solution for the initial condition

$$v(0) = +v_T \tag{1.31}$$

corresponding to the case when the mass is thrown upwards with terminal velocity. Plot the velocity as a function of time for this case in Figure 1.1.

(b) Calculate the time when the mass reaches the highest point.

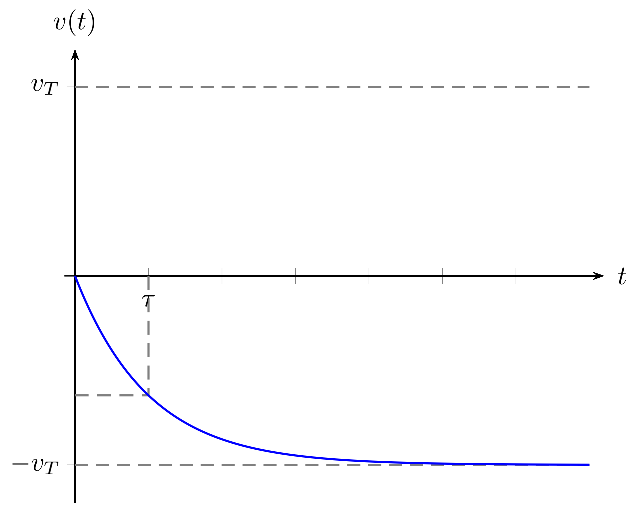

Figure 1.1: Velocity as a function of time for a mass starting from rest.

2. (**20 points.**) Consider the case when the friction force is quadratically proportional to velocity,

$$F_f = \frac{1}{2} D\rho A v^2, \tag{1.32}$$

where A is the area of crosssection of the object, ρ is the density of the medium, and D is a dimensionless drag coefficient. This should be contrasted with the case when the drag is linear in velocity. Typically, for small speeds, or when the size of the object is small, the drag force is linear in velocity. This is the case for motion in a highly viscous fluid, or for micron sized organisms in water. On the other hand, a sky diver, or a car on an interstate, experience quadratic drag forces.

   (a) For a mass m falling under uniform gravity we have the equation of motion

$$m\frac{dv}{dt} = mg - F_f. \tag{1.33}$$

   (b) Show that the terminal velocity, when $dv/dt = 0$, is given by

$$v_T = \sqrt{\frac{2mg}{D\rho A}}. \tag{1.34}$$

   (c) Solve the equation of motion for the initial condition where the particle starts from rest, $v(0) = 0$, and show that it leads to the solution

$$v(t) = v_T \frac{\left(1 - e^{-\frac{2t}{\tau}}\right)}{\left(1 + e^{-\frac{2t}{\tau}}\right)}, \tag{1.35}$$

   where $\tau = v_T/g$ sets the scale for time.

   (d) The corresponding solution for linear drag is

$$v(t) = v_T\left(1 - e^{-\frac{t}{\tau}}\right), \tag{1.36}$$

   where now $F_f = bv$ and $v_T = \frac{mg}{b}$ with $\tau = v_T/g$. Plot and compare the two velocity functions assuming the same terminal velocities.

3. (**20 points.**) Electric charge in a uniform magnetic field. Refer 20210121 video.

4. (**20 points.**) Motion of a charged particle of mass m and charge q in a uniform magnetic field $\mathbf{B}$ and a uniform electric field $\mathbf{E}$ is governed by

$$m\frac{d\mathbf{v}}{dt} = q\,\mathbf{E} + q\,\mathbf{v}\times\mathbf{B}. \tag{1.37}$$

Choose $\mathbf{B}$ along the z-axis and $\mathbf{E}$ along the y-axis,

$$\mathbf{B} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + B\,\hat{\mathbf{k}}, \tag{1.38a}$$
$$\mathbf{E} = 0\,\hat{\mathbf{i}} + E\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}. \tag{1.38b}$$

Solve this vector differential equation to determine the position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t)$ of the particle as a function of time, for initial conditions

$$\mathbf{x}(0) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{1.39a}$$
$$\mathbf{v}(0) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}. \tag{1.39b}$$

Verify that the solution is a cycloid characterized by the equations

$$x(t) = R(\omega_c t - \sin\omega_c t), \tag{1.40a}$$
$$y(t) = R(1 - \cos\omega_c t). \tag{1.40b}$$

where

$$R = \frac{E}{B\omega_c}, \qquad \omega_c = \frac{qB}{m}. \tag{1.41}$$

The particle moves as though it were a point on the rim of a wheel of radius R perfectly rolling (without sliding or slipping) with angular speed ω_c along the x-axis. It satisfies the equation of a circle of radius R whose center $(vt, R, 0)$ travels along the x-direction at constant speed v,

$$(x - vt)^2 + (y - R)^2 = R^2, \tag{1.42}$$

where $v = \omega_c R$.

Chapter 2

Calculus of variations

2.1 Functional derivative

1. (**Resource:**) The following classroom lecture from Spring 2024,

   https://youtu.be/x05ZdUxzOKo,

   serves as a good resource for functional derivative.

2. Give an account of the fundamental functional derivative

$$\frac{\delta u(x)}{\delta u(x')} = \delta(x - x'). \tag{2.1}$$

   Observe that dimensional consistency requires

$$\left[\frac{\delta}{\delta u(x)}\right] = \frac{1}{[u][x]}. \tag{2.2}$$

3. In discrete multi-variable calculus we have a function

$$f(y^i) \tag{2.3}$$

   dependent on variables

$$y^i, \qquad i = 1, 2, \ldots, \tag{2.4}$$

   such that for each i we have the derivative

$$\frac{\partial f}{\partial y^i} = \lim_{\Delta y^i \to 0} \frac{f(y^j + \Delta y^j) - f(y^j)}{\Delta y^i} \tag{2.5}$$

   evaluated in such a way that the variation in y^j is independent of a variation in y^i unless $i = j$, that is,

$$\frac{\partial y^j}{\partial y^i} = \delta_i{}^j, \tag{2.6}$$

   where $\delta_i{}^j$ is the Kronecker delta symbol.
   In continuous multi-variable calculus we have a functional

$$F[y] \tag{2.7}$$

   dependent on functions

$$y(x), \qquad x_1 < x < x_2, \tag{2.8}$$

such that for each x we have the derivative

$$\frac{\delta F[y]}{\delta y(x)} = \lim_{\Delta y(x)\to 0} \frac{F(y+\Delta y) - F(y)}{\Delta y(x)} \tag{2.9}$$

evaluated in such a way that the variation in $y(x')$ is independent of a variation in $y(x)$ unless $x = x'$, that is,

$$\frac{\delta y(x')}{\delta y(x)} = \delta(x - x'), \tag{2.10}$$

where $\delta(x - x')$ is the Dirac delta function.

4. (**20 points.**) The vector form of the fundamental functional derivative is

$$\frac{\delta \mathbf{r}(s)}{\delta \mathbf{r}(s')} = \mathbf{1}\delta(s - s'). \tag{2.11}$$

As an illustration, we evaluate the functional derivative

$$\frac{\delta}{\delta \mathbf{r}(s')} \frac{1}{r(s)}, \tag{2.12}$$

where $r(s)$ is the magnitude of the vector $\mathbf{r}(s)$, as

$$\frac{\delta}{\delta \mathbf{r}(s')} \frac{1}{r(s)} = \frac{\delta}{\delta \mathbf{r}(s')} \frac{1}{\sqrt{\mathbf{r}(s)\cdot\mathbf{r}(s)}} \tag{2.13a}$$

$$= -\frac{1}{2} \frac{2\,\mathbf{r}(s)}{(\mathbf{r}(s)\cdot\mathbf{r}(s))^{\frac{3}{2}}} \cdot \frac{\delta \mathbf{r}(s)}{\delta \mathbf{r}(s')} \tag{2.13b}$$

$$= -\frac{\mathbf{r}(s)}{r(s)^3}\delta(s - s'). \tag{2.13c}$$

2.1.1 Problems

1. (**20 points.**) The principal identity of functional differentiation is

$$\frac{\delta u(x)}{\delta u(x')} = \delta(x - x'), \tag{2.14}$$

which states that the variation in the function u at x is independent of the variation in the function u at x' unless $x = x'$. This is a generalization of the identity in multivariable calculus

$$\frac{\partial u^j}{\partial u^i} = \delta_i{}^j, \tag{2.15}$$

which states that the variables u^i and u^j are independent unless $i = j$. Using the property of δ-function,

$$\int_{-\infty}^{\infty} dx\, a(x)\delta(x - x') = a(x'), \tag{2.16}$$

derive the following identities by repeatedly differentiating by parts.

(a)

$$\int_{-\infty}^{\infty} dx\, a(x)\frac{d}{dx}\delta(x - x') = -\frac{d}{dx'}a(x') \tag{2.17}$$

(b)

$$\int_{-\infty}^{\infty} dx\, a(x)\frac{d^2}{dx^2}\delta(x - x') = +\frac{d^2}{dx'^2}a(x') \tag{2.18}$$

(c)

$$\int_{-\infty}^{\infty} dx\, a(x) \frac{d^3}{dx^3} \delta(x - x') = -\frac{d^3}{dx'^3} a(x') \tag{2.19}$$

(d)

$$\int_{-\infty}^{\infty} dx\, a(x) \frac{d^n}{dx^n} \delta(x - x') = (-1)^n \frac{d^n}{dx'^n} a(x') \tag{2.20}$$

2. (**20 points.**) Evaluate the functional derivative

$$\frac{\delta F[u]}{\delta u(x)} \tag{2.21}$$

of the following functionals, assuming no variation at the end points.

(a)

$$F[u] = \int_{x_1}^{x_2} dx\, a(x) u(x) \tag{2.22}$$

(b)

$$F[u] = \int_{x_1}^{x_2} dx\, a(x) u(x)^2 \tag{2.23}$$

(c)

$$F[u] = \int_{x_1}^{x_2} dx\, \sqrt{1 + u(x)^2} \tag{2.24}$$

(d)

$$F[u] = \int_{x_1}^{x_2} dx\, \Big[u(x) + a(x)\Big] \Big[u(x) + b(x)\Big] \tag{2.25}$$

(e)

$$F[u] = \int_{x_1}^{x_2} dx\, \frac{a(x) u(x)}{\Big[1 + b(x) u(x)\Big]} \tag{2.26}$$

3. (**20 points.**) [Refer: Gelfand and Fomin, Calculus of Variations.] Evaluate the functional derivative

$$\frac{\delta F[y]}{\delta y(x)} \tag{2.27}$$

of the following functionals, assuming no variation at the end points.

(a)

$$F[y] = \int_0^1 dx\, \frac{dy}{dx} \tag{2.28}$$

(b)

$$F[y] = \int_{x_1}^{x_2} dx\, a(x) \frac{dy(x)}{dx} \tag{2.29}$$

(c)

$$F[y] = \int_0^1 dx\, y \frac{dy}{dx} \tag{2.30}$$

(d)

$$F[y] = \int_0^1 dx\, xy \frac{dy}{dx} \tag{2.31}$$

(e)

$$F[y] = \int_a^b \frac{dx}{x^3} \left(\frac{dy}{dx} \right)^2 \tag{2.32}$$

4. (**20 points.**) Evaluate the functional derivative

$$\frac{\delta F[u]}{\delta u(x)} \tag{2.33}$$

of the following functional,

$$F[u] = \int_a^b dx\, u \sqrt{1 + \left(\frac{du}{dx} \right)^2}, \tag{2.34}$$

assuming no variation at the end points.

5. (**20 points.**) Using ab initio method, evaluate the functional derivative

$$\frac{\delta F[u]}{\delta u(x)} \tag{2.35}$$

of the following functional,

$$F[u] = \int_a^b dx \left[1 + b(x) \frac{du(x)}{dx} \right]^{\frac{3}{2}}, \tag{2.36}$$

assuming no variation at the end points.

6. (**20 points.**) Evaluate the functional derivative

$$\frac{\delta F[u]}{\delta u(x)} \tag{2.37}$$

of the following functionals, assuming no variation at the end points. Given $a(x)$ is a known function.

(a)

$$F[u] = \int_{x_1}^{x_2} dx\, a(x) \left[1 + \frac{du(x)}{dx} + \frac{d^2u(x)}{dx^2} + \frac{d^3u(x)}{dx^3} \right] \tag{2.38}$$

(b)

$$F[u] = \int_a^b dx \frac{1}{\left(1 + \frac{d^3u}{dx^3} \right)} \tag{2.39}$$

(c)

$$F[u] = \int_a^b dx\, x^5 \sqrt{1 + \frac{d^3u}{dx^3}} \tag{2.40}$$

(d)

$$F[u] = \int_a^b dx \sqrt{1 + \frac{du}{dx} + \frac{d^3u}{dx^3}} \tag{2.41}$$

7. (**20 points.**) Evaluate the functional derivative

$$\frac{\delta W[u]}{\delta u(t)} \tag{2.42}$$

of the following functionals, with u replaced with the appropriate variable, assuming no variation at the end points.

(a) Let $x(t)$ be position at time t of mass m. The action

$$W[x] = \int_{t_1}^{t_2} dt\, \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 \tag{2.43}$$

is a functional of position.

(b) Let $z(t)$ be the vertical height at time t of mass m in a uniform gravitational field g. The action

$$W[z] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \left(\frac{dz}{dt}\right)^2 - mgz\right] \tag{2.44}$$

is a functional of the vertical height.

(c) Let $x(t)$ be the stretch at time t of a spring of spring constant k attached to a mass m. The action

$$W[x] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 - \frac{1}{2} kx^2\right] \tag{2.45}$$

is a functional of the stretch.

(d) Let $r(t)$ be the radial distance at time t of mass m released from rest in a gravitational field of a planet of mass M. The action

$$W[r] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{GMm}{r}\right] \tag{2.46}$$

is a functional of the radial distance.

(e) Let $r(t)$ be the radial distance at time t of charge q_1 of mass m released from rest in an electrostatic field of another charge of charge q_2. The action

$$W[r] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \left(\frac{dr}{dt}\right)^2 - \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r}\right] \tag{2.47}$$

is a functional of the radial distance.

8. (**20 points.**) Let us investigate the fundamental identity of functional differentiation,

$$\frac{\delta f(x)}{\delta f(x')} = \delta(x - x'), \tag{2.48}$$

in the context of Fourier tranformation

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} \tilde{f}(k), \tag{2.49a}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x). \tag{2.49b}$$

Observe that the above Fourier transformation implies the δ-function representation

$$\delta(x - x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ik(x-x')}. \tag{2.50}$$

Interpreting Eq. (2.49a) as a functional in $\tilde{f}$ show that

$$\frac{\delta f(x)}{\delta \tilde{f}(k)} = \frac{1}{2\pi}\, e^{ikx}. \tag{2.51}$$

Similarly, interpreting Eq. (2.49b) as a functional in f show that

$$\frac{\delta \tilde{f}(k)}{\delta f(x)} = e^{-ikx}. \tag{2.52}$$

Using these results in the functional chain rule

$$\frac{\delta f(x)}{\delta f(x')} = \int_{-\infty}^{\infty} dk \, \frac{\delta f(x)}{\delta \tilde{f}(k)} \frac{\delta \tilde{f}(k)}{\delta f(x')} \tag{2.53}$$

obtain the fundamental identity in Eq. (2.48).

9. (**20 points.**) The eletrostatic energy of a charge distribution $\rho(\mathbf{r})$ is

$$E[\rho] = \frac{1}{2} \int d^3 r \int d^3 r' \, \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{2.54}$$

Evaluate

$$\frac{\delta^2 E}{\delta \rho(\mathbf{r})\rho(\mathbf{r}')}. \tag{2.55}$$

10. (**20 points.**) Consider the action, in terms of the Lagrangian viewpoint,

$$W[\mathbf{x}] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \left(\frac{d\mathbf{x}}{dt} \right)^2 - U(\mathbf{x}, t) \right]. \tag{2.56}$$

Assume no variation at the end points t_1 and t_2. Evaluate the functional derivative

$$\frac{\delta W}{\delta \mathbf{x}(t)}. \tag{2.57}$$

11. (**20 points.**) Consider the action, in terms of the Hamiltonian viewpoint,

$$W[\mathbf{x}, \mathbf{p}] = \int_{t_1}^{t_2} dt \left[\mathbf{p} \cdot \frac{d\mathbf{x}}{dt} - \frac{p^2}{2m} - U(\mathbf{x}, t) \right]. \tag{2.58}$$

Assume no variation at the end points t_1 and t_2. Evaluate the functional derivatives

$$\frac{\delta W}{\delta \mathbf{x}(t)} \quad \text{and} \quad \frac{\delta W}{\delta \mathbf{p}(t)}. \tag{2.59}$$

12. (**20 points.**) Consider the action, in terms of the Schwingerian viewpoint,

$$W[\mathbf{x}, \mathbf{p}, \mathbf{v}] = \int_{t_1}^{t_2} dt \left[\mathbf{p} \cdot \left(\frac{d\mathbf{x}}{dt} - \mathbf{v} \right) + \frac{1}{2} m v^2 - U(\mathbf{x}, t) \right]. \tag{2.60}$$

Assume no variation at the end points t_1 and t_2. Evaluate the functional derivatives

$$\frac{\delta W}{\delta \mathbf{x}(t)}, \quad \frac{\delta W}{\delta \mathbf{v}(t)}, \quad \text{and} \quad \frac{\delta W}{\delta \mathbf{p}(t)}. \tag{2.61}$$

13. (**40 points.**) Consider the following construction in a field theoretical setup

$$W[K] = \frac{1}{2} \int dx \int dx' K(x) \Delta(|x - x'|) K(x'), \tag{2.62}$$

where W is the action written in terms of a source function $K(x)$ and the Green's function $\Delta(|x-x'|)$. Determine the relation between the corresponding field $\phi(x)$ and the source, by evaluating the functional derivative

$$\phi(x) = \frac{\delta W}{\delta K(x)}. \tag{2.63}$$

Show that the Green's function satisfies

$$\Delta(|x-x'|) = \frac{\delta^2 W}{\delta K(x)\delta K(x')}. \tag{2.64}$$

Construct the partition function

$$Z[K] = e^{iW[K]}. \tag{2.65}$$

Show that

(a) the field satisfies

$$\phi(x) = \frac{1}{i}\frac{\delta \ln Z}{\delta K(x)} \tag{2.66}$$

(b) and the Green's function is given by

$$\Delta(|x-x'|) = \frac{1}{i}\frac{1}{Z}\frac{\delta^2 Z}{\delta K(x)\delta K(x')}\bigg|_{K=0}. \tag{2.67}$$

14. (**40 points.**) Consider the functional

$$W[x] = \int_{t_1}^{t_2} dt\, L(x,\dot{x}) \tag{2.68}$$

constructed out of the function $x = x(t)$ and its derivative $\dot{x} = dx/dt$. In particular, let

$$\frac{\partial L}{\partial t} = 0. \tag{2.69}$$

(a) Show that

$$\frac{\delta I[x]}{\delta x(t)} = \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] + \Big[\delta(t-t_2) - \delta(t-t_1)\Big]\frac{\partial L}{\partial \dot{x}}. \tag{2.70}$$

(b) Further, show that

$$\frac{\delta I[x]}{\delta x(t)} = \frac{1}{\dot{x}}\frac{d}{dt}\left(L - \dot{x}\frac{\partial L}{\partial \dot{x}}\right) + \Big[\delta(t-t_2) - \delta(t-t_1)\Big]\frac{\partial L}{\partial \dot{x}}. \tag{2.71}$$

This property used with the extremum principle, is the essence of the Beltrami identity. This also gives us a glimpse of the Legendre transform,

$$H = \dot{x}\frac{\partial L}{\partial \dot{x}} - L. \tag{2.72}$$

2.2 Fermat's principle

1. (**20 points.**) The speed of light in a medium is given in terms of the refractive index

$$n = \frac{c}{v}, \tag{2.73}$$

of the medium, where c is the speed of light in vacuum and v is the speed of light in the medium. Fermat's principle in ray optics states that a ray of light takes the path of least time between two given points. Consider a ray of light traversing a path from (x_1, y_1) to (x_2, y_2) in a stratified (layered) medium, in a plane of fixed z.

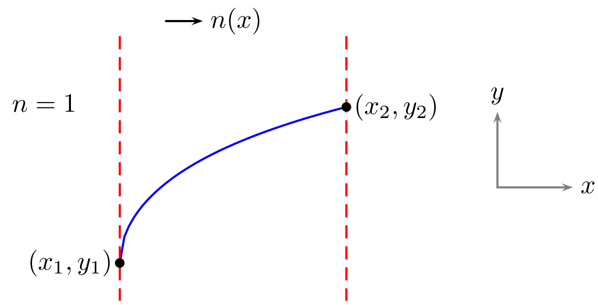

Figure 2.1: Problem 1.

(a) Show that the time taken to travel an infinitesimal distance ds is given by

$$dt = \frac{ds}{v} = \frac{n\,ds}{c}, \tag{2.74}$$

where ds in a plane of constant z is characterized by the infinitesimal statement

$$ds^2 = dx^2 + dy^2. \tag{2.75}$$

(b) Fermat's principle states that the path traversed by a ray of light from (x_1, y_1) to (x_2, y_2) is the extremal of the functional

$$T[y] = \frac{1}{c}\int_{(x_1,y_1)}^{(x_2,y_2)} n\,ds = \frac{1}{c}\int_{x_1}^{x_2} dx\, n(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{2.76}$$

(c) Since the ray of light passes through the points (x_1, y_1) and (x_2, y_2), we do not consider variations at these (end) points. Thus, show that

$$\frac{\delta T[y]}{\delta y(x)} = -\frac{1}{c}\frac{d}{dx}\left[\frac{n(x)\dfrac{dy}{dx}}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}\right]. \tag{2.77}$$

(d) Using Fermat's principle show that the differential equation for the path $y(x)$ traversed by the ray of light is

$$\frac{d}{dx}\left[\frac{n(x)\dfrac{dy}{dx}}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}\right] = 0. \tag{2.78}$$

In terms of the function $\theta(x)$ defined using the relation

$$\tan\theta(x) = \frac{dy}{dx} \tag{2.79}$$

express the differential equation in the form

$$\frac{d}{dx}\Big[n(x)\sin\theta(x)\Big] = 0. \tag{2.80}$$

Thus, derive Snell's law for refraction,

$$n(x)\sin\theta(x) = \eta, \tag{2.81}$$

where η is a constant.

2. (**20 points.**) Snell's law for refraction for stratified (layered) medium states that

$$n(x)\sin\theta(x) = \eta, \tag{2.82}$$

where η is a constant. Show that Snell's law can be rewritten in the form

$$\frac{dy}{dx} = \frac{\eta}{\sqrt{n(x)^2 - \eta^2}}. \tag{2.83}$$

(a) Let us consider a medium with refractive index ($x_1 = a$)

$$n(x) = \begin{cases} 1, & x < a, \\ \dfrac{x}{a}, & a < x. \end{cases} \tag{2.84}$$

Solve the corresponding differential equation, by substituting $x = \eta\, a \cosh t$, to obtain

$$y(x) - y_0 = \eta\, a \cosh^{-1}\left(\frac{1}{\eta}\frac{x}{a}\right), \qquad a < x. \tag{2.85}$$

Thus, the path in this medium satisfies the equation of a catenary. It is also useful to express the solution in terms of the logarithm as

$$y(x) - y_0 = \eta\, a \ln\left[\frac{1}{\eta}\frac{x}{a} + \sqrt{\left(\frac{1}{\eta}\frac{x}{a}\right)^2 - 1}\right], \qquad a < x. \tag{2.86}$$

(b) For initial conditions, ($x_1 = a$,)

$$y(x_1) = y_1 \qquad \text{and} \qquad \left.\frac{dy}{dx}\right|_{x=x_1} = y_1' \tag{2.87}$$

show that integration constants are determined as

$$y_0 = y_1 - \eta\, a \ln\left[\frac{1}{\eta} + \sqrt{\frac{1}{\eta^2} - 1}\right], \qquad \text{and} \qquad \eta = \frac{y_1'}{\sqrt{1 + {y_1'}^2}}. \tag{2.88}$$

Thus, write the solution as

$$y(x) - y_1 = \eta\, a \ln\left[\frac{\dfrac{1}{\eta}\dfrac{x}{a} + \sqrt{\dfrac{1}{\eta^2}\dfrac{x^2}{a^2} - 1}}{\dfrac{1}{\eta} + \sqrt{\dfrac{1}{\eta^2} - 1}}\right], \qquad a < x. \tag{2.89}$$

(c) For the special case $y_1 = 0$ and $y_1' \to \infty$ show that $\eta = 1$ and

$$y(x) = a \ln\left[\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right], \qquad a < x. \tag{2.90}$$

(d) Evaluate the total time taken for light to go from (x_1, y_1) to $(x, y(x))$.
Solution:

$$T = \frac{a}{c}\frac{\eta^2}{2}\left[\ln\left(\frac{x + \sqrt{x^2 - \eta^2 a^2}}{a + \sqrt{a^2 - \eta^2 a^2}}\right) + \frac{x}{\eta\, a}\sqrt{\frac{x^2}{\eta^2 a^2} - 1} - \frac{1}{\eta}\sqrt{\frac{1}{\eta^2} - 1}\right], \qquad a < x. \tag{2.91}$$

3. (**20 points.**) Snell's law for refraction for stratified (layered) medium states that

$$n(x)\sin\theta(x) = \eta, \tag{2.92}$$

where η is a constant. Show that Snell's law can be rewritten in the form

$$\frac{dy}{dx} = \frac{\eta}{\sqrt{n(x)^2 - \eta^2}}. \tag{2.93}$$

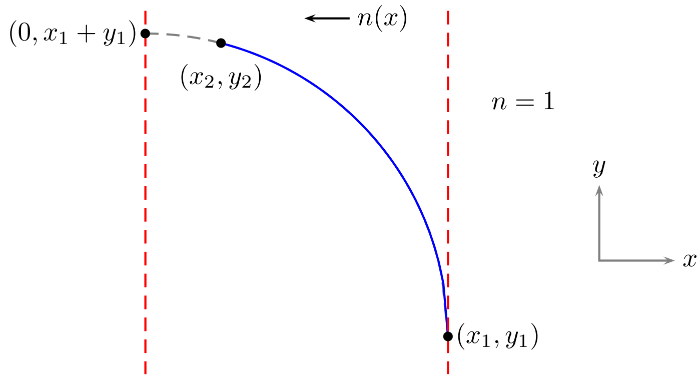

Figure 2.2: Problem 3.

(a) Let us consider a medium with refractive index ($x_1 = a$)

$$n(x) = \begin{cases} \dfrac{a}{x}, & 0 < x < a, \\ 1, & a < x. \end{cases} \tag{2.94}$$

Solve the corresponding differential equation to obtain

$$y(x) - y_0 = \frac{1}{\eta}\left[\sqrt{a^2 - \eta^2 x^2} - \sqrt{a^2 - \eta^2 a^2}\right], \qquad x < a. \tag{2.95}$$

The path in this medium satisfies the equation of a circle. Determine the radius of the circle to be a/η and the location of the center to be $(0, y_0 - a\sqrt{(1/\eta^2) - 1})$. For initial conditions

$$y(x_1) = y_1 \qquad \text{and} \qquad \left.\frac{dy}{dx}\right|_{x=x_1} = y_1' \tag{2.96}$$

show that the integration constants are determined to be

$$y_0 = y_1 \qquad \text{and} \qquad \eta = \frac{y_1'}{\sqrt{1 + {y_1'}^2}}. \tag{2.97}$$

For the special case when $y_1 = 0$ and $y_1' \to \infty$ show that $\eta = 1$ and

$$y(x) = \sqrt{a^2 - x^2}, \qquad x < a. \tag{2.98}$$

Evaluate the total time taken for light to go from $(x_1 = a, y_1 = 0)$ to $(x_2 = 0, y_2 = a)$.

(b) **To do:** Check for a sign in the solution for y. Further, should $y_1' \to -\infty$? Refer solutions to MT-01 in Spring 2020.
Solution:
The time taken is given by

$$cT = -\int_a^0 dx\, n(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2} = a \lim_{\delta\to 0}\int_\delta^1 \frac{dt}{t}\frac{1}{\sqrt{1-t^2}}. \tag{2.99}$$

The negative sign in the expression corresponds to velocity being negative. This yields

$$cT = -a\ln\left(\frac{\delta}{1+\sqrt{1-\delta^2}}\right) \sim a(\ln 2 - \ln\delta), \tag{2.100}$$

which diverges logarithmically. Thus, the light takes infinite time to reach the point $(0, a)$ from $(a, 0)$.

2.3 Geodesics on surfaces

1. (**20 points.**) Let us prove the intuitively obvious statement that the curve of shortest distance going through two points (x_1, y_1) and (x_2, y_2) in a plane, the geodesics of a plane, is a straight line passing through the two points.

   (a) The distance between two points in a plane is characterized by the infinitesimal statement

   $$ds^2 = dx^2 + dy^2. \tag{2.101}$$

   (b) The geodesic is the extremal of the functional

   $$l[y] = \int_{(x_1,y_1)}^{(x_2,y_2)} ds = \int_{x_1}^{x_2} dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}. \tag{2.102}$$

   (c) Since the curve passes through the points (x_1, y_1) and (x_2, y_2), we have no variations at these (end) points. Thus, show that

   $$\frac{\delta l[y]}{\delta y(x)} = -\frac{d}{dx}\left[\frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right]. \tag{2.103}$$

   (d) Using the extremum principle

   $$\frac{\delta l[y]}{\delta y(x)} = 0 \tag{2.104}$$

   show that the differential equation for the geodesic is

   $$\frac{dy}{dx} = c, \tag{2.105}$$

   where c is a contant.

   (e) Solve the differential equation to identify the equation of a straight line in a plane. Find c.

2. (**20 points.**) Let the distance between two points in a plane be characterized by the infinitesimal statement

   $$ds^4 = dx^4 + dy^4. \tag{2.106}$$

   The geodesic is the extremal of the functional

   $$l[y] = \int_{(x_1,y_1)}^{(x_2,y_2)} ds. \tag{2.107}$$

   Find the geodesics. Recognize them.

3. (**20 points.**) Find the geodesics on the surface of a circular cylinder. Identify these curves. Hint: To have a visual perception of these geodesics it helps to note that a cylinder can be mapped (or cut open) into a plane.

   (a) The distance between two points on the surface of a cylinder of radius a is characterized by the infinitesimal statement
   $$ds^2 = a^2 d\phi^2 + dz^2. \tag{2.108}$$

   (b) The geodesic is the extremal of the functional
   $$l[z] = \int_{(\phi_1,z_1)}^{(\phi_2,z_2)} ds = \int_{\phi_1}^{\phi_2} a d\phi \sqrt{1 + \left(\frac{1}{a}\frac{dz}{d\phi}\right)^2}. \tag{2.109}$$

   (c) Since the curve passes through the points (z_1, ϕ_1) and (z_2, ϕ_2) we have no variations on the end points. Thus, show that
   $$\frac{\delta l[z]}{\delta z(\phi)} = -\frac{d}{d\phi}\left[\frac{\frac{1}{a}\frac{dz}{d\phi}}{\sqrt{1 + \left(\frac{1}{a}\frac{dz}{d\phi}\right)^2}}\right]. \tag{2.110}$$

   (d) Using the extremum principle
   $$\frac{\delta l[z]}{\delta z(\phi)} = 0 \tag{2.111}$$
   show that the differential equation for the geodesic is
   $$\frac{1}{a}\frac{dz}{d\phi} = c, \tag{2.112}$$
   where c is a contant.

   (e) Solve the differential equation. Identify the curve described by the solution to be a helix. Illustrate a particular curve using a diagram.
   Solution: $z = ca\phi + c_2$. Helix.

4. (**20 points.**) The geodesics on the surface of a circular cylinder of radius a are helices,
   $$z = c_1 a\phi + c_2, \tag{2.113}$$
   where c_1 and c_2 are arbitrary constants and the distance between two points is characterized by the infinitesimal statement
   $$ds^2 = a^2 d\phi^2 + dz^2. \tag{2.114}$$
   The geodesic is the extremal of the functional
   $$l[z] = \int_{(\phi_1,z_1)}^{(\phi_2,z_2)} ds = \int_{\phi_1}^{\phi_2} a d\phi \sqrt{1 + \left(\frac{1}{a}\frac{dz}{d\phi}\right)^2}. \tag{2.115}$$
   Find the length of a geodesic passing through the points $(\phi_1 = 0, z_1 = 0)$ and $(\phi_2 = \pi, z_2 = \pi a)$.

5. (**20 points.**) Show that the geodesics on a spherical surface are great circles, that is, circles whose centers lie at the center of the sphere.

   (a) The distance between two points on the surface of a sphere of radius a is characterized by the infinitesimal statement
   $$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2. \tag{2.116}$$

(b) The geodesic is the extremal of the functional

$$l[\phi] = \int_{(\theta_1,\phi_1)}^{(\theta_2,\phi_2)} ds = \int_{\theta_1}^{\theta_2} a d\theta \sqrt{1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2}. \tag{2.117}$$

(c) Since the curve passes through the points (θ_1, ϕ_1) and (θ_2, ϕ_2) we have no variations on the end points. Thus, show that

$$\frac{1}{a}\frac{\delta l[\phi]}{\delta\phi(\theta)} = -\frac{d}{d\theta}\left[\frac{\sin^2\theta \dfrac{d\phi}{d\theta}}{\sqrt{1 + \sin^2\theta \left(\dfrac{d\phi}{d\theta}\right)^2}}\right]. \tag{2.118}$$

(d) Using the extremum principle

$$\frac{\delta l[\phi]}{\delta\phi(\theta)} = 0 \tag{2.119}$$

show that the differential equation for the geodesic is

$$\frac{d\phi}{d\theta} = \frac{c}{\sin\theta\sqrt{\sin^2\theta - c^2}}, \tag{2.120}$$

where c is an arbitrary constant.

(e) Solve the differential equation in Eq. (2.120) to obtain the equation of the geodesic as

$$\sin(\phi + \phi_0) + \bar{c}\cot\theta = 0, \tag{2.121}$$

where $\bar{c} = c/\sqrt{1 - c^2}$, and ϕ_0 is a constant of integration.
Hint: Integrate the equation in Eq. (2.120) to write

$$\phi + \phi_0 = \int \frac{c\, d\theta}{\sin\theta\sqrt{\sin^2\theta - c^2}}, \tag{2.122}$$

where ϕ_0 is a constant of integration. Then, substitute

$$t = \frac{c}{\sqrt{1 - c^2}}\cot\theta \tag{2.123}$$

and complete the resulting integral to obtain

$$\phi + \phi_0 = -\sin^{-1} t. \tag{2.124}$$

(f) Expand the sine function in Eq. (2.121) and express it in the form

$$\sin\phi_0 \sin\theta\cos\phi + \cos\phi_0 \sin\theta\sin\phi + \bar{c}\cos\theta = 0. \tag{2.125}$$

Interpret this to be an equation of a plane passing through the origin by recognizing the form

$$(\mathbf{r} - \mathbf{r}_0)\cdot\hat{\mathbf{n}} = 0, \tag{2.126}$$

where

$$\mathbf{r} = a\sin\theta\cos\phi\,\hat{\mathbf{x}} + a\sin\theta\sin\phi\,\hat{\mathbf{y}} + a\cos\theta\,\hat{\mathbf{z}}, \tag{2.127a}$$

$$\mathbf{r}_0 = 0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{2.127b}$$

$$\hat{\mathbf{n}} = \sqrt{1 - c^2}\sin\phi_0\,\hat{\mathbf{x}} + \sqrt{1 - c^2}\cos\phi_0\,\hat{\mathbf{y}} + \frac{c}{\sqrt{1 - c^2}}\,\hat{\mathbf{z}}. \tag{2.127c}$$

This plane passes through the origin because $\mathbf{r}_0 = 0$. The vector $\hat{\mathbf{n}}$ is a constant unit vector determined by the integration constants c and ϕ_0, which are determined by the two original points (θ_1, ϕ_1) and (θ_2, ϕ_2). The vector $\mathbf{r}$ is an arbitrary point on the sphere of radius a. The condition in Eq. (2.126) selects those points on the sphere that are on the plane that is perpendicular to $\hat{\mathbf{n}}$ and that passes through the origin (center of sphere.) Thus, these points lie on a great circle.

6. (**20 points.**) Find the geodesics on the surface of a cone with opening angle θ.
Hint: To have a visual perception of these geodesics it helps to note that a cone can be mapped (or cut open) into a plane.
Solution:

$$r(\phi) = \frac{r_0}{\sin(\sin\theta(\phi - \phi_0))}. \tag{2.128}$$

7. (**20 points.**) Find the geodesics on the surface of a circular cylinder.
Solution: Helix.

$$z(\phi) = c_1\phi + c_2. \tag{2.129}$$

2.4 Brachistochrone on surfaces

8. (**60 points.**) Consider a rope of uniform mass density $\lambda = dm/ds$ hanging from two points, (x_1, y_1) and (x_2, y_2), as shown in Figure 2.3. The gravitational potential energy of an infinitely tiny element of this

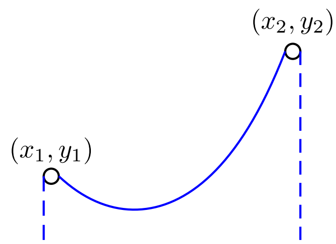

Figure 2.3: Problem 8.

rope at point (x, y) is given by

$$dU = dm\, gy = \lambda g ds\, y, \tag{2.130}$$

where

$$ds^2 = dx^2 + dy^2. \tag{2.131}$$

A catenary is the curve that the rope assumes, that minimizes the total potential energy of the rope.

(a) Show that the total potential energy U of the rope hanging between points x_1 and x_2 is given by

$$U[x] = \lambda g \int_{(x_1,y_1)}^{(x_2,y_2)} y ds = \lambda g \int_{y_1}^{y_2} dy\, y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}. \tag{2.132}$$

(b) Since the curve passes through the points (x_1, y_1) and (x_2, y_2), we have no variations at these (end) points. Thus, show that

$$\frac{\delta U[x]}{\delta x(y)} = -\lambda g \frac{d}{dy}\left[y \frac{\frac{dx}{dy}}{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}} \right]. \tag{2.133}$$

(c) Using the extremum principle show that the differential equation for the catenary is

$$\frac{dx}{dy} = \frac{a}{\sqrt{y^2 - a^2}}, \tag{2.134}$$

where a is an integration contant.

(d) Show that integration of the differential equation yields the equation of the catenary

$$y = a \cosh \frac{x - x_0}{a}, \tag{2.135}$$

where x_0 is another integration constant.

(e) For the case $y_1 = y_2$ we have

$$\frac{y_1}{a} = \cosh \frac{x_1 - x_0}{a}, \tag{2.136a}$$

$$\frac{y_2}{a} = \cosh \frac{x_2 - x_0}{a}, \tag{2.136b}$$

which leads to, assuming $x_1 \neq x_2$,

$$x_0 = \frac{x_1 + x_2}{2}. \tag{2.137}$$

Identify x_0 in Figure 2.3. Next, derive

$$\frac{y_1}{a} = \frac{y_2}{a} = \cosh \frac{x_2 - x_1}{2a}, \tag{2.138}$$

which, in principle, determines a. However, this is a transcendental equation in a and does not allow exact evaluation of a in closed form and one depends on numerical solutions. Observe that, if $x = x_0$ in Eq. (2.135), then $y = a$. Identify a in Figure 2.3.

9. (**20 points.**) A catenary is the curve that an idealized hanging chain assumes under its own weight when supported only at its ends in a uniform gravitational field. It is the curve $y(x)$ that minimizes the potential energy U of the hanging chain

$$U = \int dU = \int dm\, gy = \frac{Mg}{P} \int y ds, \tag{2.139}$$

where M is the mass of the uniform chain, P is the length of the chain, g is the acceleration due to gravity. Let us assume the two end points of the chain are at the same height. A catenary is given by

$$y = a \cosh \frac{x}{a}, \tag{2.140}$$

where the parameter a, an integration constant, characterizes the catenary. Find the relation between the parameter a, the perimeter length P of the chain, and the height y_0.

(a) Determine the perimeter length P of the hanging chain using

$$P = \int_{-x_0}^{x_0} ds. \tag{2.141}$$

(b) Show that the relation between the parameter a, the perimeter length P of the chain, and the height y_0 in Figure 2.4 is given by

$$a = \sqrt{y_0^2 - \left(\frac{P}{2}\right)^2} \tag{2.142}$$

(c) Show that the distance x_0 is given by

$$x_0 = a \cosh^{-1} \frac{y_0}{a} = a \ln \left(\frac{y_0}{a} \pm \frac{P}{2a} \right). \tag{2.143}$$

Show that in the limit $a \to 0$ $x_0 \to 0$. This corresponds to the case $y_0 \to P/2$.

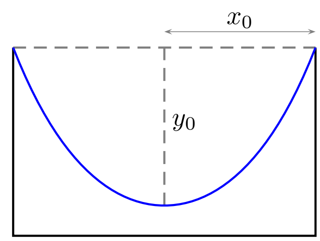

Figure 2.4: Problem 9.

10. (**20 points.**) A catenary is described by

$$y = a \cosh\left(\frac{x - x_0}{a}\right), \tag{2.144}$$

where constants a and x_0 are determined by the position of the end points (x_1, y_1) and (x_2, y_2). Let us choose $x_0 = 0$ and $a = 1$ such that

$$y = \cosh x, \tag{2.145}$$

where x and y are dimensionless variables.

(a) Using series expansion show that

$$\cosh x = 1 + \frac{x^2}{2} + \ldots. \tag{2.146}$$

(b) The parabola $y = 1 + x^2/2$ hugs the catenary at $x = 0$, but it does not pass through the end points. Consider the parabola

$$y = 1 + \frac{\alpha}{2} x^2 \tag{2.147}$$

that also hugs the parabola at $x = 0$. Determine α such that this parabola passes through $x = \pm 1$. Choose this parabola to be an approximation for the catenary. Plot this parabola and the catenary in the same plot for $-1 < x < 1$ and estimate the maximum deviation (with sign) in this approximation. Does the parabola sag below the catenary, or is it the other way around.
Solution: $\alpha = (\cosh(1) - 1) \sim 1.08616$. Maximum deviation is -0.010 (about 1%). Thus, the parabola sags below the catenary.

11. (**20 points.**) (Based on Problem 7 in Chapter 2 of Goldstein, 2nd edition.) Catenoid: A rope of uniform linear mass density and indefinite length passes freely over pulleys at equal heights $y_1 = y_2$, above the surface of Earth, with horizontal distance $x_2 - x_1$ between them. (Assume uniform gravitational field.) Determine the curve followed by the rope hanging between the pulleys. Compare (using plots) the catenoid and a parabola.

12. (**20 points.**) Write a brief summary on the Isoperimetric problem, and problem of minimum surface of revolution. For example, refer Goldstein, Chapter 2.
A related note: A gyroid is an infinitely connected triply periodic minimal surface discovered by Alan Schoen, who is a retired faculty of the Math department in SIUC and currently a resident of Carbondale.

Chapter 3

Stationary action principle (with no variations at boundary)

3.1 Euler-Lagrange equations

1. (**30 points.**) The motion of a particle of mass m near the Earth's surface is described by

$$\frac{d}{dt}(mv) = -mg, \tag{3.1}$$

where $v = dz/dt$ is the velocity in the upward z direction.

(a) Find the Lagrangian for this system that implies the equation of motion of Eq. (3.1) using the principle of stationary action.

(b) Determine the canonical momentum for this system

(c) Determine the Hamilton $H(p, z)$ for this system.

(d) Determine the Hamilton equations of motion.

2. (**30 points.**) The motion of a particle of mass m undergoing simple harmonic motion is described by

$$\frac{d}{dt}(mv) = -kx, \tag{3.2}$$

where $v = dx/dt$ is the velocity in the x direction.

(a) Find the Lagrangian for this system that implies the equation of motion of Eq. (3.2) using the principle of stationary action.

(b) Determine the canonical momentum for this system

(c) Determine the Hamiltonian $H(p, x)$ for this system.

(d) Determine the Hamilton equations of motion.

3. (**30 points.**) Given the Lagrangian

$$L_1(z, v) = \frac{1}{2}mv^2 - mgz, \tag{3.3}$$

find the equation of motion. Next, given another Lagrangian

$$L_2(z, v) = \frac{1}{2}mv^2 - mgz + bvz, \tag{3.4}$$

find the equation of motion. Analyze and justify.

4. (**30 points.**) A non-relativistic charged particle of charge q and mass m in the presence of a known electric and magnetic field is described by

$$\frac{d}{dt}(m\mathbf{v}) = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}. \tag{3.5}$$

   (a) Using

$$\mathbf{B} = \nabla\times\mathbf{A}, \tag{3.6a}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \tag{3.6b}$$

   find the Lagrangian for this system, that implies the equation of motion of Eq. (3.5), to be

$$L(\mathbf{x},\mathbf{v},t) = \frac{1}{2}mv^2 - q\phi + q\mathbf{v}\cdot\mathbf{A}, \tag{3.7}$$

   using the principle of stationary action.

   (b) Determine the canonical momentum for this system

   (c) Determine the Hamiltonian $H(\mathbf{x},\mathbf{p},t)$ for this system to be

$$H(\mathbf{x},\mathbf{p},t) = \frac{1}{2m}(\mathbf{p}-q\mathbf{A})^2 + q\phi. \tag{3.8}$$

5. (**30 points.**) A relativistic charged particle of charge q and mass m in the presence of a known electric and magnetic field is described by

$$\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}. \tag{3.9}$$

   (a) Find the Lagrangian for this system, that implies the equation of motion of Eq. (3.9), to be

$$L(\mathbf{x},\mathbf{v},t) = -mc^2\sqrt{1-\frac{v^2}{c^2}} - q\phi + q\mathbf{v}\cdot\mathbf{A}, \tag{3.10}$$

   using the principle of stationary action.

   (b) Determine the canonical momentum for this system

   (c) Determine the Hamilton $H(\mathbf{r},\mathbf{p})$ for this system to be

$$H(\mathbf{x},\mathbf{p},t) = \sqrt{m^2c^4 + (\mathbf{p}-q\mathbf{A})^2c^2} + q\phi. \tag{3.11}$$

6. (**20 points.**) Verify, by substitution in Eqs. (3.6), that a plausible scalar and vector potential for constant (uniform in space and time) electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are

$$\phi = -\mathbf{r}\cdot\mathbf{E}, \tag{3.12a}$$

$$\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}. \tag{3.12b}$$

   Thus, show that

$$q\phi - q\mathbf{v}\cdot\mathbf{A} = -\mathbf{d}\cdot\mathbf{E} - \boldsymbol{\mu}\cdot\mathbf{B}, \tag{3.13}$$

   where $\mathbf{d} = q\mathbf{r}$ is the electric dipole moment and $\boldsymbol{\mu} = \frac{q}{2}\mathbf{r}\times\mathbf{v} = \frac{q}{2m}\mathbf{L}$, with $\mathbf{L} = \mathbf{r}\times m\mathbf{v}$, is the magnetic dipole moment.

7. (**20 points.**) The Hamiltonian

$$H(\mathbf{x},\mathbf{p},t) = \frac{1}{2m}\left(\mathbf{p} - q\mathbf{A}\right)^2 + q\phi \tag{3.14}$$

describes a non-relativistic particle of charge q and mass m in the presence of a known electric and magnetic field. Find the Hamiltonian equations of motion to be

$$\frac{d\mathbf{x}}{dt} = \frac{1}{m}\left(\mathbf{p} - q\mathbf{A}\right), \tag{3.15a}$$

$$\begin{aligned}\frac{d\mathbf{p}}{dt} &= -q\boldsymbol{\nabla}\phi + q(\boldsymbol{\nabla}\mathbf{A})\cdot\frac{(\mathbf{p} - q\mathbf{A})}{m}.\\ &= -\boldsymbol{\nabla}\Big[q\phi - q\mathbf{v}\cdot\mathbf{A}\Big]\end{aligned} \tag{3.15b}$$

Further, show that the above equations, in conjunction, imples the Lorentz force equation

$$\frac{d}{dt}\left(m\mathbf{v}\right) = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}. \tag{3.16}$$

8. (**20 points.**) Consider the action,

   (a) in terms of the Lagrangian viewpoint,

   $$W[\mathbf{x}] = \int_{t_1}^{t_2} dt\left[\frac{1}{2}m\left(\frac{d\mathbf{x}}{dt}\right)^2 - U(\mathbf{x},t)\right]. \tag{3.17}$$

   Assume no variation at the end points t_1 and t_2. Evaluate the functional derivative

   $$\frac{\delta W}{\delta\mathbf{x}(t)}. \tag{3.18}$$

   (b) in terms of the Hamiltonian viewpoint,

   $$W[\mathbf{x},\mathbf{p}] = \int_{t_1}^{t_2} dt\left[\mathbf{p}\cdot\frac{d\mathbf{x}}{dt} - \frac{p^2}{2m} - U(\mathbf{x},t)\right]. \tag{3.19}$$

   Assume no variation at the end points t_1 and t_2. Evaluate the functional derivatives

   $$\frac{\delta W}{\delta\mathbf{x}(t)} \quad\text{and}\quad \frac{\delta W}{\delta\mathbf{p}(t)}. \tag{3.20}$$

   (c) in terms of the Schwingerian viewpoint,

   $$W[\mathbf{x},\mathbf{p},\mathbf{v}] = \int_{t_1}^{t_2} dt\left[\mathbf{p}\cdot\left(\frac{d\mathbf{x}}{dt} - \mathbf{v}\right) + \frac{1}{2}mv^2 - U(\mathbf{x},t)\right]. \tag{3.21}$$

   Assume no variation at the end points t_1 and t_2. Evaluate the functional derivatives

   $$\frac{\delta W}{\delta\mathbf{x}(t)},\quad \frac{\delta W}{\delta\mathbf{v}(t)},\quad\text{and}\quad \frac{\delta W}{\delta\mathbf{p}(t)}. \tag{3.22}$$

9. (**20 points.**) Consider a (time independent) Hamiltonian

$$H = H(x,p), \tag{3.23}$$

which satisfies the Hamilton equations of motion

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \tag{3.24a}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x}. \tag{3.24b}$$

(a) Recollect that the Lagrangian, which will temporarily be called the x-Lagrangian here, is defined by the construction

$$L_x(x, \dot{x}) = p\dot{x} - H(x, p). \tag{3.25}$$

Starting from Eq. (3.25) derive

$$\frac{\partial L_x}{\partial x} = -\frac{\partial H}{\partial x}, \tag{3.26a}$$

$$\frac{\partial L_x}{\partial \dot{x}} = p, \tag{3.26b}$$

$$\frac{\partial L_x}{\partial p} = \dot{x} - \frac{\partial H}{\partial p}. \tag{3.26c}$$

Using the Hamilton equations of motion, Eqs. (3.24), in Eqs. (3.26) we have the equations governing the x-Lagrangian to be

$$\frac{\partial L_x}{\partial p} = 0, \tag{3.27a}$$

$$\frac{\partial L_x}{\partial \dot{x}} = p, \tag{3.27b}$$

$$\frac{d}{dt}\frac{\partial L_x}{\partial \dot{x}} = \frac{\partial L_x}{\partial x}. \tag{3.27c}$$

(b) Now, define the p-Lagrangian using the construction

$$L_p(p, \dot{p}) = -x\dot{p} - H(x, p). \tag{3.28}$$

The opposite sign in the construction of the p-Lagrangian is motivated by the action principle, which does not care for a total derivative, refer Schwinger. Starting from Eq. (3.28) derive

$$\frac{\partial L_p}{\partial p} = -\frac{\partial H}{\partial p}, \tag{3.29a}$$

$$\frac{\partial L_p}{\partial \dot{p}} = -x, \tag{3.29b}$$

$$\frac{\partial L_p}{\partial x} = -\dot{p} - \frac{\partial H}{\partial x}. \tag{3.29c}$$

Using the Hamilton equations of motion, Eqs. (3.24), in Eqs. (3.29) we have the equations governing the p-Lagrangian to be

$$\frac{\partial L_p}{\partial x} = 0, \tag{3.30a}$$

$$\frac{\partial L_p}{\partial \dot{p}} = -x, \tag{3.30b}$$

$$\frac{d}{dt}\frac{\partial L_p}{\partial \dot{p}} = \frac{\partial L_p}{\partial p}. \tag{3.30c}$$

(c) Illustrate the above for a harmonic oscillator. Show that

$$L_x(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2, \tag{3.31a}$$

$$H(x, p) = \frac{p^2}{2m} + \frac{1}{2}kx^2, \tag{3.31b}$$

$$L_p(p, \dot{p}) = \frac{\dot{p}^2}{2k} + \frac{p^2}{2m}. \tag{3.31c}$$

10. (**20 points.**) Consider the Lagrangian for the unbounded quartic potential

$$L(s, \dot{s}) = \frac{1}{2}\dot{s}^2 + g s^4. \tag{3.32}$$

Using the transformation on the complex plane

$$s = -2i\sqrt{1 + ix} \tag{3.33}$$

obtain the Lagrangian

$$L_x(x, \dot{x}) = \frac{1}{2}\frac{\dot{x}^2}{(1 + ix)} + 16g(1 + ix)^2 \tag{3.34}$$

and show that

$$p = \frac{\partial L_x}{\partial \dot{x}} = \frac{\dot{x}}{(1 + ix)}. \tag{3.35}$$

Construct the Hamiltonian

$$H(x, p) = p\dot{x} - L_x(x, \dot{x}) \tag{3.36}$$

and show that

$$H(x, p) = \frac{1}{2}p^2(1 + ix) - 16g(1 + ix)^2. \tag{3.37}$$

Derive the Hamilton equations of motion

$$\dot{x} = p(1 + ix), \tag{3.38}$$

$$-\frac{\dot{p}}{i} = \frac{p^2}{2} - 32g(1 + ix). \tag{3.39}$$

Further, derive

$$\ddot{p} = -\frac{1}{2}p^3. \tag{3.40}$$

Construct the p-Lagrangian

$$L_p(p, \dot{p}) = -x\dot{p} - H(x, p) \tag{3.41}$$

and show that

$$L_p(p, \dot{p}) = \frac{1}{32g}\left[\frac{\dot{p}^2}{2} - \frac{p^4}{8}\right] \tag{3.42}$$

upto a total derivative.

11. (**20 points.**) Construct the Hamiltonian from the Lagrangian

$$L(x, \dot{x}) = \frac{1}{2}\frac{\dot{x}^2}{(1 + ix)} + 16g(1 + ix)^2 \tag{3.43}$$

and show that the equation of motion can be expressed in the form

$$\ddot{p} = -\alpha p^3. \tag{3.44}$$

Find α. Here

$$p = \frac{\partial L}{\partial \dot{x}} \tag{3.45}$$

is the canonical momentum.

12. (**20 points.**) Consider the Hamiltonian
$$H = H(\mathbf{r}, \mathbf{p}, t), \tag{3.46}$$
which satisfies the Hamilton equations of motion
$$\frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \tag{3.47a}$$
$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{r}}. \tag{3.47b}$$
The p-Lagrangian is constructed usng the definition
$$L_p = -\mathbf{r} \cdot \frac{d\mathbf{p}}{dt} - H(\mathbf{r}, \mathbf{p}, t). \tag{3.48}$$
Investigate the dependence of the p-Lagrangian on the variable $\mathbf{r}$ by evaluating the partial derivative with respect to it. That is, evaluate
$$\frac{\partial L_p}{\partial \mathbf{r}}. \tag{3.49}$$

13. (**20 points.**) Given a Lagrangian L, the Hamiltonian H is given by
$$H = \mathbf{p} \cdot \frac{d\mathbf{r}}{dt} - L, \tag{3.50}$$
where $\mathbf{p}$ is the canonical momentum. Evaluate
$$\frac{\partial H}{\partial \mathbf{v}}, \tag{3.51}$$
where $\mathbf{v}$ stands for $d\mathbf{r}/dt$.

14. (**20 points.**) Given a time-independent Hamiltonian
$$H = H(x, p) \tag{3.52}$$
and the corresponding Hamilton equations of motion, show that
$$\frac{dH}{dt} = \alpha, \tag{3.53}$$
where α is a number. Evaluate α. What is the physical interpretation?

15. (**20 points.**) Given
$$U = U(S, V) \tag{3.54}$$
and
$$dU = T\,dS - P\,dV \tag{3.55}$$
and
$$F = U - TS, \tag{3.56}$$
evaluate
$$\frac{\partial F}{\partial S}. \tag{3.57}$$

16. (**20 points.**) (Refer Goldstein, 2nd edition, Chapter 1 Problem 8.) As a consequence of the Hamilton's stationary action principle, the equations of motion for a system can be expressed as Euler-Lagrange equations,
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, \tag{3.58}$$

in terms of a Lagrangian $L(x, \dot{x}, t)$. Show that the Lagrangian for a system is not unique. In particular, show that if $L(x, \dot{x}, t)$ satisfies the Euler-Lagrange equation then

$$L'(x, \dot{x}, t) = L(x, \dot{x}, t) + \frac{dF(x, t)}{dt}, \tag{3.59}$$

where $F(x, t)$ is any arbitrary differentiable function, also satisfies the Euler-Lagrange equation.

17. (**20 points.**) Consider the four-vector $x^\alpha = (ct, \mathbf{x})$. Here $\alpha = 0, 1, 2, 3$, such that $x^0 = ct$ and x^i are the three components of vector $\mathbf{x}$. The proper time s, that remains invariant under a Lorentz transformation, satisfies

$$-ds^2 = -c^2 dt^2 + d\mathbf{x} \cdot d\mathbf{x}. \tag{3.60}$$

Thus, derive the relation

$$\frac{1}{c}\frac{ds}{dt} = \sqrt{1 - \frac{v^2}{c^2}}, \tag{3.61}$$

where $\mathbf{v} = d\mathbf{x}/dt$. The energy E and momentum $\mathbf{p}$ of a particle of mass m is defined as

$$mc^2 \frac{dx^\alpha}{ds} = (E, \mathbf{p}c). \tag{3.62}$$

Find the explicit expressions for E and $\mathbf{p}$ in terms of $\mathbf{v}$, c, and m. Show that

$$\frac{dx^\alpha}{ds}\frac{dx_\alpha}{ds} = -1, \tag{3.63}$$

and use this to derive $E^2 = p^2c^2 + m^2c^4$.

18. (**20 points.**) The motion of an electric charge of mass m in a time-independent magnetic field, (in the absence of an electric field,) is described by the equation of motion

$$\frac{d}{dt}(m\mathbf{v}) = q\mathbf{v} \times \mathbf{B}. \tag{3.64}$$

Recall, the electric ad magnetic field is given in terms of (gauge dependent) electric and magnetic potentials as

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{3.65}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{3.66}$$

The electric potential and magnetic vector potentials are by construction independent of velocity. For time-independent magnetic field, (in the absence of an electric field,) we have

$$\boldsymbol{\nabla}\phi = 0, \qquad \frac{\partial \mathbf{A}}{\partial t} = 0, \tag{3.67}$$

and

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{3.68}$$

(a) Starting from the equation of motion derive

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = 0. \tag{3.69}$$

Thus, kinetic energy is conserved. We shall see that the Hamiltonian for this configuration is numerically equal to the kinetic energy. We emphasize that to satisfy the Hamilton equations of motion the Hamiltonian has to be written in terms of position of momentum, not velocity.

(b) Starting from the equation of motion derive

$$\frac{d}{dt}(m\mathbf{v} + q\mathbf{A}) = \boldsymbol{\nabla}(q\mathbf{v}\cdot\mathbf{A}). \tag{3.70}$$

Compare with the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}} = \frac{\partial L}{\partial \mathbf{r}} \tag{3.71}$$

and identify

$$\frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + q\mathbf{A}, \tag{3.72a}$$

$$\frac{\partial L}{\partial \mathbf{r}} = \boldsymbol{\nabla}(q\mathbf{v}\cdot\mathbf{A}). \tag{3.72b}$$

Thus, find a Lagrangian

$$L(\mathbf{r},\mathbf{v}) = \frac{1}{2}mv^2 + q\mathbf{v}\cdot\mathbf{A}. \tag{3.73}$$

(c) Show that the canonical momentum is given by

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + q\mathbf{A}. \tag{3.74}$$

(d) Construct the Hamiltonian using

$$H(\mathbf{r},\mathbf{p}) = \mathbf{p}\cdot\mathbf{v} - L \tag{3.75}$$

and show that

$$H(\mathbf{r},\mathbf{p}) = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m}. \tag{3.76}$$

Show that the Hamiltonian is equal to kinetic energy, that is,

$$H = \frac{1}{2}mv^2. \tag{3.77}$$

(e) Derive the Hamilton equations of motion,

$$\frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{p}} = \frac{(\mathbf{p} - q\mathbf{A})}{m}, \tag{3.78}$$

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{r}} = \boldsymbol{\nabla}(q\mathbf{v}\cdot\mathbf{A}). \tag{3.79}$$

(f) For a constant magnetic field show that

$$\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r} \tag{3.80}$$

is a magnetic vector potential upto a gauge. That is, evaluate the curl of $\mathbf{A}$. Show that

$$-q\mathbf{v}\cdot\mathbf{A} = -\frac{q}{2m}\mathbf{B}\cdot\mathbf{L}, \tag{3.81}$$

where $\mathbf{L} = \mathbf{r}\times(m\mathbf{v})$ is an angular momentum.

19. (**20 points.**) A two-body system constituting of two masses m_1 and m_2, described by an interaction energy that depends only on the difference in the position of the two bodies

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \tag{3.82}$$

is described by the Lagrangian

$$L(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2) = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - U(\mathbf{r}_1 - \mathbf{r}_2). \tag{3.83}$$

(a) Derive the canonical momentums,

$$\mathbf{p}_1 = m_1\mathbf{v}_1 \quad \text{and} \quad \mathbf{p}_2 = m_2\mathbf{v}_2. \tag{3.84}$$

Find the Euler-Lagrange equations,

$$\frac{d}{dt}(m_1\mathbf{v}_1) = -\boldsymbol{\nabla}_1 U(\mathbf{r}_1 - \mathbf{r}_2), \tag{3.85a}$$

$$\frac{d}{dt}(m_2\mathbf{v}_2) = \boldsymbol{\nabla}_2 U(\mathbf{r}_1 - \mathbf{r}_2). \tag{3.85b}$$

Construct the Hamiltonian

$$H(\mathbf{r}_1, \mathbf{p}_1, \mathbf{r}_2, \mathbf{p}_2) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + U(\mathbf{r}_1 - \mathbf{r}_2) \tag{3.86}$$

and derive the Hamilton equations of motion.

(b) In terms of coordinates representing the position of center of mass

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \tag{3.87}$$

and the relative position of the masses

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{3.88}$$

show that

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{m_1 + m_2}\mathbf{r}, \tag{3.89a}$$

$$\mathbf{r}_2 = \mathbf{R} - \frac{m_1}{m_1 + m_2}\mathbf{r}. \tag{3.89b}$$

In terms of the respective velocities

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} \quad \text{and} \quad \mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{3.90}$$

derive

$$\mathbf{v}_1 = \mathbf{V} + \frac{m_2}{m_1 + m_2}\mathbf{v}, \tag{3.91a}$$

$$\mathbf{v}_2 = \mathbf{V} - \frac{m_1}{m_1 + m_2}\mathbf{v}. \tag{3.91b}$$

(c) In terms of these coordinates show that

$$L(\mathbf{r}, \mathbf{v}, \mathbf{R}, \mathbf{V}) = \frac{1}{2}MV^2 + \frac{1}{2}\mu v^2 - U(\mathbf{r}), \tag{3.92}$$

where M is the total mass and μ is the reduced mass,

$$M = m_1 + m_2 \quad \text{and} \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{3.93}$$

Observe that the reduced mass is always smaller than the smaller of the two masses. Observe that the Lagrangian is independent of $\mathbf{R}$ even though it depends on the associated velocity $\mathbf{V}$. Such a coordinate is called a cyclic coordinate. Determine the canonical momentums,

$$\mathbf{P} = \frac{\partial L}{\partial \mathbf{V}} = M\mathbf{V}, \tag{3.94a}$$

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = \mu\mathbf{v}. \tag{3.94b}$$

Find the Euler-Lagrange equations of motion,

$$\frac{d}{dt}(M\mathbf{V}) = 0, \tag{3.95a}$$

$$\frac{d}{dt}(\mu\mathbf{v}) = -\boldsymbol{\nabla}U. \tag{3.95b}$$

Here the momentum associated with the cyclic coordinate $\mathbf{R}$ is conserved. The canonical momentum associated to a cyclic coordinate is always conserved.

(d) Show that

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \tag{3.96}$$

is the toal momentum of the system and

$$\mathbf{p} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{m_1 + m_2} \tag{3.97}$$

can be defined as the relative momentum. Construct the Hamiltonian using the Legendre transform,

$$H(\mathbf{r}, \mathbf{p}, \mathbf{R}, \mathbf{P}) = \frac{P^2}{2M} + \frac{p^2}{2\mu} + U(\mathbf{r}). \tag{3.98}$$

Derive the Hamilton equations of motion

$$\frac{d\mathbf{R}}{dt} = \frac{\mathbf{P}}{M}, \tag{3.99a}$$

$$\frac{d\mathbf{P}}{dt} = 0, \tag{3.99b}$$

and

$$\frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{\mu}, \tag{3.100a}$$

$$\frac{d\mathbf{p}}{dt} = -\boldsymbol{\nabla}U. \tag{3.100b}$$

20. (**20 points.**) The Kepler problem is described by the Lagrangian

$$L(r, \phi, \dot{r}, \dot{\phi}) = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 + \frac{\alpha}{r}, \tag{3.101}$$

where the first term is the contribution to the kinetic energy of the particle with reduced mass μ due to radial velocity $\dot{r}$, the second term is the contribution to the kinetic energy due to it's tangential velocity $\dot{\phi}$, and the third term is the negative of the gravitational potential energy between masses m_1 and m_2. Here $\alpha = G\mu M$. (Show that $\mu M = m_1 m_2$.) Show that the canonical momentum in the radial direction and the associated force are

$$p_r = \frac{\partial L}{\partial \dot{r}} = \mu\dot{r}, \tag{3.102a}$$

$$F_r = \frac{\partial L}{\partial r} = \mu r\dot{\phi}^2 - \frac{\alpha}{r^2}, \tag{3.102b}$$

respectively. Thus, derive the equation for the radial motion to be

$$\frac{d}{dt}\mu\dot{r} = \mu r\dot{\phi}^2 - \frac{\alpha}{r^2}, \tag{3.103}$$

where the second term on the right is the gravitational force of attraction and the first term on the right is the centrifugal force due to the continuous change in the direction of tangential velocity. Show that the canonical momentum in the tangential direction, the angular momentum, and the associated canonical force, the torque, are

$$L_z = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \dot{\phi}, \tag{3.104a}$$

$$F_\phi = \frac{\partial L}{\partial \phi} = 0, \tag{3.104b}$$

so that the angular momentum is a constant of motion,

$$\frac{d}{dt} L_z = 0. \tag{3.105}$$

(a) Using the conservation of angular momentum in Eq. (3.104a) to replace $\dot{\phi}$ in the equation of motion in Eq. (3.103) derive

$$\frac{d}{dt}\mu \dot{r} = \frac{L_z^2}{\mu r^3} - \frac{\alpha}{r^2}, \tag{3.106}$$

such that we can write

$$\frac{d}{dt}\mu \dot{r} = \frac{\partial}{\partial r}\left(-\frac{L_z^2}{2\mu r^2} + \frac{\alpha}{r}\right). \tag{3.107}$$

Multiplying by $\dot{r}$ on both sides gives

$$\dot{r}\frac{d}{dt}\mu \dot{r} = \frac{dr}{dt}\frac{\partial}{\partial r}\left(-\frac{L_z^2}{2\mu r^2} + \frac{\alpha}{r}\right) \tag{3.108}$$

which can be written in the form

$$\frac{d}{dt}\left(\frac{1}{2}\mu \dot{r}^2 + \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r}\right) = 0 \tag{3.109}$$

and is interpreted as the statement of conservation of energy.

(b) Find the error in the following steps. Using the conservation of angular momentum in Eq. (3.104a) to replace $\dot{\phi}$ in the Lagrangian in Eq. (3.101) derive

$$L(r, \dot{r}) = \frac{1}{2}\mu \dot{r}^2 + \frac{L_z^2}{2\mu r^2} + \frac{\alpha}{r} \tag{3.110}$$

and derive the equation of motion

$$\frac{d}{dt}\mu \dot{r} = \frac{\partial}{\partial r}\left(\frac{L_z^2}{2\mu r^2} + \frac{\alpha}{r}\right). \tag{3.111}$$

Thus, derive the statement of conservation of energy as

$$\frac{d}{dt}\left(\frac{1}{2}\mu \dot{r}^2 - \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r}\right) = 0 \tag{3.112}$$

with a wrong sign. The lesson learnt is that substituting an equation of motion to replace a variable inside the Lagrangian demotes it from the status of the Lagrangian.

21. (**20 points.**) Describe the motion corresponding to the Hamiltonian

$$H(\mathbf{r}, \mathbf{p}) = \frac{p^2}{2m} + \frac{1}{2}k(x^2 - y^2), \tag{3.113}$$

where $\mathbf{r} = \hat{\mathbf{i}}\,x + \hat{\mathbf{j}}\,y + \hat{\mathbf{k}}\,z$ is position $\mathbf{p}$ is the associated momentum, and m and k are constants. In particular, plot the trajectory of motion for the the initial conditions

$$\mathbf{r}(0) = \hat{\mathbf{i}}\,0 + \hat{\mathbf{j}}\,R + \hat{\mathbf{k}}\,0, \tag{3.114a}$$

$$\mathbf{v}(0) = \hat{\mathbf{i}}\,\omega R + \hat{\mathbf{j}}\,0 + \hat{\mathbf{k}}\,0, \tag{3.114b}$$

where $\omega = \sqrt{k/m}$ and R is a non-zero length.

22. (**20 points.**) [Based on Landau and Lifshitz. Section 7.] A particle of mass m moving with velocity $\mathbf{v}_1$ leaves a half-space in which the potential energy is a constant U_1 and enters another in which the potential energy is a different constant $U_2 > U_1$.

(a) Force is the manifestation of the system trying to attain minimum energy. Draw the velocity vector $\mathbf{v}_2$ in Fig. 3.1 that satisfies these conditions. Does it deflect away from normal or towards the normal?

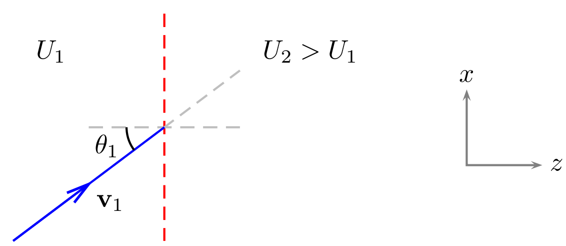

Figure 3.1: Problem 22.

(b) The potential energy can be described by

$$U(\mathbf{r}) = \begin{cases} U_1, & z < a, \\ U_2, & a < z. \end{cases} \tag{3.115}$$

In terms of the Heavyside step function

$$\theta(z) = \begin{cases} 0, & z < 0, \\ 1, & 0 < z, \end{cases} \tag{3.116}$$

show that the potential energy can be expressed in the form

$$U(\mathbf{r}) = U_1 + (U_2 - U_1)\theta(z - a). \tag{3.117}$$

(c) Show that a suitable Lagrangian for the motion is

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}mv^2 - U_1 - (U_2 - U_1)\theta(z - a). \tag{3.118}$$

Derive the relations

$$\frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v}, \tag{3.119a}$$

$$\frac{\partial L}{\partial \mathbf{r}} = -\hat{\mathbf{z}}\,(U_2 - U_1)\,\delta(z - a). \tag{3.119b}$$

Recall that the derivative of Heaviside step function is a δ-function. Thus, derive the equation of motion

$$\frac{d}{dt}m\mathbf{v} = -\hat{\mathbf{z}}\,(U_2 - U_1)\,\delta(z - a). \tag{3.120}$$

(d) Show that the momentum in the plane perpendicular to $\hat{\mathbf{z}}$ is conserved. Refer Fig. 3.1. That is,

$$v_1 \sin\theta_1 = v_2 \sin\theta_2. \tag{3.121}$$

Show that the energy is conserved. That is,

$$\frac{1}{2} m v_1^2 + U_1 = \frac{1}{2} m v_2^2 + U_2. \tag{3.122}$$

Thus, derive the measure of deflection at the interface to be given by

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{1 - \frac{2(U_2 - U_1)}{m v_1^2}}. \tag{3.123}$$

Chapter 4

Lagrangian mechanics

4.1 Constraint equations and Lagrangian

1. (**20 points.**) A mass m_1 is forced to move on a vertical circle of radius R with uniform angular speed ω. Another mass m_2 is connected to mass m_1 using a massless rod of length a, such that it is a simple pendulum with respect to mass m_1. Motion of both the masses is constrained to be in a vertical plane in a uniform gravitational field.

   (a) Write the Lagrangian for the system.

   (b) Determine the equation of motion for the system.

   (c) Give physical interpretation of each term in the equation of motion.

2. (**20 points.**) A system, characterized by the parameters ω, α, and β, and the dynamical parameter θ, is described by the equation of motion

$$\ddot{\theta} + \omega^2 \sin\theta + \alpha\ddot{\theta}\cos\theta + \beta\dot{\theta}^2 \sin\theta = 0. \tag{4.1}$$

   Write the above equation of motion in the small angle approximation, to the leading order in θ.

3. (**20 points.**) A mass m slides down a frictionless ramp that is inclined at an angle θ with respect to the horizontal. See Fig. 4.1. Assume uniform gravity g in the vertical downward direction.

   (a) What is the constraint in the variables.

   (b) In terms of a suitable dynamical variable write a Lagrangian that describes the motion of the mass.

   (c) Find the equations of motion from the Lagrangian.

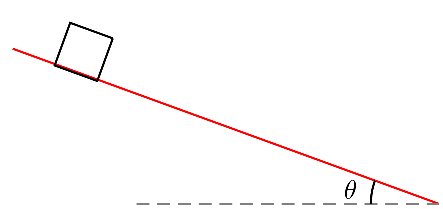

Figure 4.1: Problem 3.

4. (**20 points.**) The Atwood machine consists of two masses m_1 and m_2 connected by a massless (inextensible) string passing over a massless pulley. See Figure 4.2. Massless pulley implies that tension in the string on both sides of the pulley is the same, say T. Further, the string being inextensible implies that the magnitude of the accelerations of both the masses are the same. Let $m_2 > m_1$.

   (a) What is the constraint in the variables.

   (b) In terms of a suitable dynamical variable write a Lagrangian that describes the motion of the mass.

   (c) Find the equations of motion from the Lagrangian.

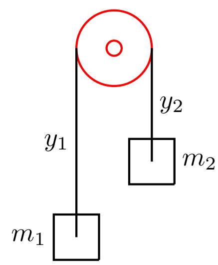

Figure 4.2: Problem 4.

5. (**20 points.**) A pendulum consists of a mass m_2 hanging from a pivot by a massless string of length a. The pivot, in general, has mass m_1, but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a horizontal rod. See Figure 5. For simplification, and at loss of generality, let us chose the motion of the pendulum in a vertical plane containing the rod.

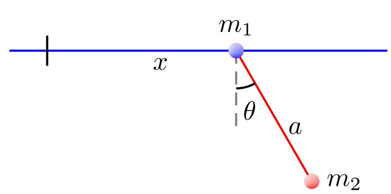

Figure 4.3: Problem 5.

   (a) Determine the Lagrangian for the system to be

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_2 a^2 \dot{\theta}^2 + m_2 a \dot{x} \dot{\theta} \cos\theta + m_2 g a \cos\theta. \tag{4.2}$$

   (b) Evaluate the following derivatives and give physical interpretations of each of these.

$$\frac{\partial L}{\partial \dot{x}} = m_2 \dot{x} + m_2 a \dot{\theta} \cos\theta, \tag{4.3a}$$

$$\frac{\partial L}{\partial x} = 0, \tag{4.3b}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_2 a^2 \dot{\theta} + m_2 a \dot{x} \cos\theta, \tag{4.3c}$$

$$\frac{\partial L}{\partial \theta} = -m_2 a \dot{x} \dot{\theta} \sin\theta - m_2 g a \sin\theta. \tag{4.3d}$$

(c) Determine the equations of motion for the system. Express them in the form

$$\ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta = 0, \tag{4.4a}$$
$$a\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0. \tag{4.4b}$$

Observe that, like in the case of simple pendulum, the motion is independent of the mass m_2 when $m_1 = 0$.

(d) In the small angle approximation show that the equations of motion reduce to

$$\ddot{x} + a\ddot{\theta} = 0, \tag{4.5a}$$
$$a\ddot{\theta} + \ddot{x} + g\theta = 0. \tag{4.5b}$$

Determine the solution to be given by

$$\theta = 0 \quad \text{and} \quad \ddot{x} = 0. \tag{4.6}$$

Interpret this solution.

(e) The solution $\theta = 0$ seems to be too restrictive. Will this system not allow $\theta \neq 0$? To investigate this, let us not restrict to the small angle approximation. Rewrite Eqs. (4.4), using Eq. (4.4a) in Eq. (4.4b), as

$$\ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta = 0, \tag{4.7a}$$
$$\sin\theta\Big[a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta + g\Big] = 0. \tag{4.7b}$$

In this form we immediately observe that $\theta = 0$ is a solution. However, it is not the only solution. Towards interpretting Eqs. (4.7) let us identify the coordinates of the center of mass of the m_1-m_2 system,

$$(m_1 + m_2)x_{\text{cm}} = m_1 x + m_2(x + a\sin\theta), \tag{4.8a}$$
$$(m_1 + m_2)y_{\text{cm}} = -m_2 a\cos\theta, \tag{4.8b}$$

which for $m_1 = 0$ are the coordinates of the mass m_2,

$$x_{\text{cm}} = x + a\sin\theta, \tag{4.9a}$$
$$y_{\text{cm}} = -a\cos\theta. \tag{4.9b}$$

Show that

$$\dot{x}_{\text{cm}} = \dot{x} + a\dot{\theta}\cos\theta, \tag{4.10a}$$
$$\dot{y}_{\text{cm}} = a\dot{\theta}\sin\theta, \tag{4.10b}$$

and

$$\ddot{x}_{\text{cm}} = \ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta, \tag{4.11a}$$
$$\ddot{y}_{\text{cm}} = a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta. \tag{4.11b}$$

Comparing Eqs. (4.7) and Eqs. (4.11) we learn that

$$\ddot{x}_{\text{cm}} = 0, \tag{4.12a}$$
$$\sin\theta\Big[\ddot{y}_{\text{cm}} + g\Big] = 0. \tag{4.12b}$$

Thus, $\ddot{y}_{\text{cm}} = -g$ is the more general solution, and $\theta = 0$ is a trivial solution.

(f) Let us analyse the system for initial conditions: $\theta(0) = \theta_0$, $\dot\theta(0) = 0$, $\dot x(0) = 0$. Show that for this case $\dot x_{\rm cm}(0) = 0$ and

$$a(\cos\theta - \cos\theta_0) = \frac{1}{2} g t^2. \tag{4.13}$$

Plot θ as a function of time t. Interpret this solution.

(g) **To do:** The interpretation does not seem satisfactory. Is $m_1 = 0$ physical here?

6. (**20 points.**) A pendulum consists of a mass m_2 hanging from a pivot by a massless string of length a_2. The pivot, in general, has mass m_1, but, for simplification let $m_1 = 0$. Let the pivot be constrained to move on a frictionless hoop of radius a_1. See Figure 6. For simplification, and at loss of generality, let us chose the motion of the pendulum in the plane containing the hoop.

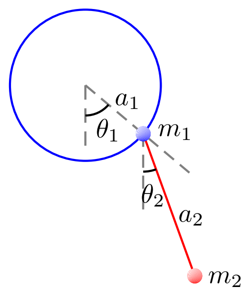

Figure 4.4: Problem 6.

(a) Determine the Lagrangian for the system to be

$$L(\theta_1, \dot\theta_1, \theta_2, \dot\theta_2) = \frac{1}{2} m_2 a_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 a_2^2 \dot\theta_2^2 + m_2 a_1 a_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1 - \theta_2) + m_2 g a_1 \cos\theta_1 + m_2 g a_2 \cos\theta_2. \tag{4.14}$$

(b) Evaluate the following derivatives and give physical interpretations of each of these.

$$\frac{\partial L}{\partial \dot\theta_1} = m_2 a_1^2 \dot\theta_1 + m_2 a_1 a_2 \dot\theta_2 \cos(\theta_1 - \theta_2), \tag{4.15a}$$

$$\frac{\partial L}{\partial \theta_1} = -m_2 a_1 a_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - m_2 g a_1 \sin\theta_1, \tag{4.15b}$$

$$\frac{\partial L}{\partial \dot\theta_2} = m_2 a_2^2 \dot\theta_2 + m_2 a_1 a_2 \dot\theta_1 \cos(\theta_1 - \theta_2), \tag{4.15c}$$

$$\frac{\partial L}{\partial \theta_2} = m_2 a_1 a_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - m_2 g a_2 \sin\theta_2. \tag{4.15d}$$

(c) Determine the equations of motion for the system. Express them in the form

$$\ddot\theta_1 + \omega_1^2 \sin\theta_1 + \frac{1}{\beta} \ddot\theta_2 \cos(\theta_1 - \theta_2) + \frac{1}{\beta} \dot\theta_2^2 \sin(\theta_1 - \theta_2) = 0, \tag{4.16a}$$

$$\ddot\theta_2 + \omega_2^2 \sin\theta_2 + \beta \ddot\theta_1 \cos(\theta_1 - \theta_2) - \beta \dot\theta_1^2 \sin(\theta_1 - \theta_2) = 0, \tag{4.16b}$$

where

$$\omega_1^2 = \frac{g}{a_1}, \quad \omega_2^2 = \frac{g}{a_2}, \quad \beta = \frac{a_1}{a_2} = \frac{\omega_2^2}{\omega_1^2}. \tag{4.17}$$

Note that β is not an independent parameter. Also, observe that, like in the case of simple pendulum, the motion is independent of the mass m_2 when $m_1 = 0$.

(d) In the small angle approximation show that the equations of motion reduce to

$$\ddot{\theta}_1 + \omega_1^2 \theta_1 + \frac{1}{\beta} \ddot{\theta}_2 = 0, \tag{4.18a}$$

$$\ddot{\theta}_2 + \omega_2^2 \theta_2 + \beta \ddot{\theta}_1 = 0. \tag{4.18b}$$

(e) Determine the solution for the initial conditions

$$\theta_1(0) = \theta_2(0) = \theta_{20}, \quad \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0. \tag{4.19}$$

Interpret and expound your solution.

7. (**20 points.**) Consider the coplanar double pendulum in Figure 7.

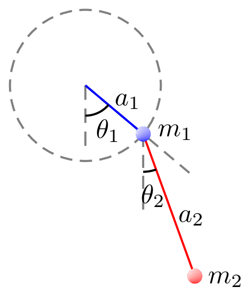

Figure 4.5: Problem 7.

(a) Write the Lagrangian for the system. In particular, show that the Lagrangian can be expressed in the form

$$L = L_1 + L_2 + L_{\text{int}}, \tag{4.20}$$

where

$$L_1 = \frac{1}{2}(m_1 + m_2) a_1^2 \dot{\theta}_1^2 + (m_1 + m_2) g a_1 \cos\theta_1, \tag{4.21a}$$

$$L_2 = \frac{1}{2} m_2 a_2^2 \dot{\theta}_2^2 + m_2 g a_2 \cos\theta_2, \tag{4.21b}$$

$$L_{\text{int}} = m_2 a_1 a_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2). \tag{4.21c}$$

(b) Determine the equations of motion for the system. Express them in the form

$$(m_1 + m_2) a_1 \ddot{\theta}_1 + (m_1 + m_2) g \sin\theta_1 + m_2 a_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 a_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = 0, \tag{4.22a}$$

$$a_2 \ddot{\theta}_2 + g \sin\theta_2 + a_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - a_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) = 0. \tag{4.22b}$$

(c) In the small angle approximation show that the equations of motion reduce to

$$\ddot{\theta}_1 + \omega_1^2 \theta_1 + \frac{\alpha}{\beta} \ddot{\theta}_2 = 0, \tag{4.23a}$$

$$\ddot{\theta}_2 + \omega_2^2 \theta_2 + \beta \ddot{\theta}_1 = 0, \tag{4.23b}$$

where

$$\omega_1^2 = \frac{g}{a_1}, \quad \omega_2^2 = \frac{g}{a_2}, \quad \alpha = \frac{m_2}{m_1 + m_2}, \quad \beta = \frac{a_1}{a_2} = \frac{\omega_2^2}{\omega_1^2}. \tag{4.24}$$

Note that $0 \leq \alpha \leq 1$.

(d) Determine the solution for the initial conditions

$$\theta_1(0) = 0, \quad \theta_2(0) = 0, \quad \dot{\theta}_1(0) = 0, \quad \dot{\theta}_2(0) = \omega_0, \tag{4.25}$$

for $\alpha = 1/2$ and $\beta = 1$.

4.2 Small angle approximation and normal modes

1. (**20 points.**) Consider the differential equation

$$\ddot{x}(t) = -\omega_1^2 x(t), \tag{4.26}$$

where dot denotes differentiation with respect to time, in conjunction with a suitable initial condition.

(a) Using Fourier transform

$$x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} \tilde{x}(\omega), \tag{4.27a}$$

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} x(t), \tag{4.27b}$$

show that $\tilde{x}(\omega)$ satisfies the algebraic equation

$$-\omega^2 \tilde{x}(\omega) = -\omega_1^2 \tilde{x}(\omega). \tag{4.28}$$

Observe that we can arrive at this equation using the transcription,

$$\frac{\partial}{\partial t} \to -i\omega, \tag{4.29a}$$

$$x(t) \to \tilde{x}(\omega), \tag{4.29b}$$

in the original differential equation. Thus, the algebraic equation for $\tilde{x}(\omega)$ is

$$(\omega^2 - \omega_1^2)\tilde{x}(\omega) = 0. \tag{4.30}$$

(b) The solution to the above algebraic equation can be expressed in the form

$$\tilde{x}(\omega) = \tilde{\alpha}(\omega)\delta(\omega^2 - \omega_1^2), \tag{4.31}$$

where $\tilde{\alpha}(\omega)$ is to be determined. Using the property of δ-functions show that

$$\tilde{x}(\omega) = \frac{\tilde{\alpha}(\omega_1)}{2\omega_1}\delta(\omega - \omega_1) + \frac{\tilde{\alpha}(-\omega_1)}{2\omega_1}\delta(\omega + \omega_1). \tag{4.32}$$

(c) Using Fourier transform evaluate

$$x(t) = \frac{1}{2\pi}\frac{\tilde{\alpha}(\omega_1)}{2\omega_1}e^{-i\omega_1 t} + \frac{1}{2\pi}\frac{\tilde{\alpha}(-\omega_1)}{2\omega_1}e^{i\omega_1 t}. \tag{4.33}$$

In terms of numbers

$$A_1 = \frac{1}{2\pi}\frac{\tilde{\alpha}(\omega_1)}{2\omega_1}, \tag{4.34a}$$

$$B_1 = \frac{1}{2\pi}\frac{\tilde{\alpha}(-\omega_1)}{2\omega_1}, \tag{4.34b}$$

express the solution in the form

$$x(t) = A_1 e^{-i\omega_1 t} + B_1 e^{i\omega_1 t}. \tag{4.35}$$

The numbers A_1 and B_1 are determined from initial conditions. For example, show that for initial conditions $x(0) = A$ and $\dot{x}(0) = 0$ the solution is

$$x(t) = A\cos\omega_1 t. \tag{4.36}$$

2. (**20 points.**) Two masses m_1 and m_2 are constrained to be on the x axis, but are free to move on the axis. They are connected by a spring of spring constant k. Negelect the gravitational force between them. Determine the normal modes of vibration for this system.

   (a) Let x_1 and $x_2 > x_1$ be the positions of the masses and a be the equilibrium length of the spring. Show that the equations of motion for the two masses are

   $$\ddot{x}_1 = +\omega_1^2(x_2 - x_1 - a), \tag{4.37a}$$
   $$\ddot{x}_2 = -\omega_2^2(x_2 - x_1 - a), \tag{4.37b}$$

   where $\omega_1^2 = k/m_1$ and $\omega_2^2 = k/m_2$. The change in length of the spring from equilibrium length a is

   $$x = x_2 - x_1 - a. \tag{4.38}$$

   Thus, show that

   $$\ddot{x} = \ddot{x}_2 - \ddot{x}_1. \tag{4.39}$$

   Thus, derive

   $$\ddot{x} = -(\omega_1^2 + \omega_2^2)x \tag{4.40}$$

   and conclude that the normal modes are

   $$\omega^2 = \omega_1^2 + \omega_2^2. \tag{4.41}$$

   (b) Show that the Lagrangian for the system is

   $$L(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k(x_2 - x_1 - a)^2. \tag{4.42}$$

   Derive the Euler-Lagrange equations of motion for x_1 and x_2. Then, introduce the transformations

   $$(m_1 + m_2)x_{\rm cm} = m_1x_1 + m_2x_2, \tag{4.43a}$$
   $$x = x_2 - x_1 - a, \tag{4.43b}$$

   and obtain the Lagrangian

   $$L(x_{\rm cm}, x, \dot{x}_{\rm cm}, \dot{x}) = \frac{1}{2}M\dot{x}_{\rm cm}^2 + \frac{1}{2}\mu\dot{x}^2 - \frac{1}{2}kx^2, \tag{4.44}$$

   where $M = m_1 + m_2$ and

   $$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{4.45}$$

   Derive the Euler-Lagrange equations for $x_{\rm cm}$ and x. Describe the motion. Determine the normal modes.

3. (**20 points.**) Find the normal modes for the system described by the Lagrangian

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}\mathbf{v}^T \cdot \mathbf{M} \cdot \mathbf{v} - \frac{1}{2}\mathbf{r}^T \cdot \mathbf{K} \cdot \mathbf{r}, \tag{4.46}$$

where $\mathbf{r} = (x_1, x_2)$ is a position vector on a plane, $\mathbf{v} = \dot{\mathbf{r}}$ is velocity, and T in superscript denotes transpose. Matrices $\mathbf{M}$ and $\mathbf{K}$ are

$$\mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad \text{and} \quad \mathbf{K} = \begin{pmatrix} k_1 & k_3 \\ k_3 & k_2 \end{pmatrix}. \tag{4.47}$$

4. (**20 points.**) Consider the set of differential equations

$$\ddot{x}_1(t) + \omega_1^2 x_1(t) = \omega_3^2 x_2(t), \tag{4.48a}$$
$$\ddot{x}_2(t) + \omega_2^2 x_2(t) = \omega_3^2 x_1(t), \tag{4.48b}$$

where dot denotes differentiation with respect to time, in conjunction with suitable initial conditions.

(a) Using Fourier transform show that $\tilde{x}_1(\omega)$ and $\tilde{x}_2(\omega)$ satisfy the algebraic equations

$$(\omega_1^2 - \omega^2)\tilde{x}_1(\omega) - \omega_3^2\tilde{x}_2(\omega) = 0, \tag{4.49a}$$
$$-\omega_3^2\tilde{x}_1(\omega) + (\omega_2^2 - \omega^2)\tilde{x}_2(\omega) = 0. \tag{4.49b}$$

Observe that they decouple for $\omega_3 = 0$. The explicit nature of the coupling is brought out by writing the solutions, $\tilde{x}_1(\omega)$ and $\tilde{x}_2(\omega)$, in the form

$$\tilde{x}_1(\omega) = \frac{\omega_3^2}{(\omega_1^2 - \omega^2)}\tilde{x}_2(\omega), \tag{4.50a}$$
$$\tilde{x}_2(\omega) = \frac{\omega_3^2}{(\omega_2^2 - \omega^2)}\tilde{x}_1(\omega). \tag{4.50b}$$

Using the two solutions in conjunction show that the solutions satisfy

$$(\omega - \lambda_1)(\omega + \lambda_1)(\omega - \lambda_2)(\omega + \lambda_2)\tilde{x}_1(\omega) = 0, \tag{4.51a}$$
$$(\omega - \lambda_1)(\omega + \lambda_1)(\omega - \lambda_2)(\omega + \lambda_2)\tilde{x}_2(\omega) = 0, \tag{4.51b}$$

where $\pm\lambda_1$ and $\pm\lambda_2$ are roots of the quartic equation

$$(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) - \omega_3^4 = 0. \tag{4.52}$$

Evaluate the roots for $\omega_2^2 > \omega_1^2$ to be

$$\lambda_2^2 = \frac{(\omega_2^2 + \omega_1^2)}{2} + \frac{1}{2}\sqrt{(\omega_2^2 - \omega_1^2)^2 + 4\omega_3^4}, \tag{4.53a}$$
$$\lambda_1^2 = \frac{(\omega_2^2 + \omega_1^2)}{2} - \frac{1}{2}\sqrt{(\omega_2^2 - \omega_1^2)^2 + 4\omega_3^4}, \tag{4.53b}$$

and express them in the form

$$\lambda_2^2 = \omega_2^2 + (\mu^2 - \Delta^2), \tag{4.54a}$$
$$\lambda_1^2 = \omega_1^2 - (\mu^2 - \Delta^2), \tag{4.54b}$$

where

$$\Delta^2 = \frac{(\omega_2^2 - \omega_1^2)}{2} \tag{4.55}$$

and

$$\mu^2 = \sqrt{\Delta^4 + \omega_3^4}. \tag{4.56}$$

Determine the normal modes λ_1 and λ_2 for $\omega_3 = 0$.

(b) Derive the following. The difference in the square of roots,

$$\lambda_2^2 - \lambda_1^2 = 2\mu^2, \tag{4.57}$$

and the change in the normal modes due to coupling,

$$\omega_1^2 - \lambda_1^2 = (\mu^2 - \Delta^2), \tag{4.58a}$$
$$\omega_1^2 - \lambda_2^2 = -(\mu^2 + \Delta^2), \tag{4.58b}$$

and

$$\omega_2^2 - \lambda_1^2 = (\mu^2 + \Delta^2), \tag{4.59a}$$
$$\omega_2^2 - \lambda_2^2 = -(\mu^2 - \Delta^2). \tag{4.59b}$$

Using the above relations together with

$$\omega_3^2 = \sqrt{(\mu^2+\Delta^2)(\mu^2-\Delta^2)} \tag{4.60}$$

derive

$$\frac{\omega_3^2}{(\omega_1^2-\lambda_1^2)} = \sqrt{\frac{\mu^2+\Delta^2}{\mu^2-\Delta^2}} = -\frac{\omega_3^2}{(\omega_2^2-\lambda_2^2)}, \tag{4.61a}$$

$$\frac{\omega_3^2}{(\omega_2^2-\lambda_1^2)} = \sqrt{\frac{\mu^2-\Delta^2}{\mu^2+\Delta^2}} = -\frac{\omega_3^2}{(\omega_1^2-\lambda_2^2)}. \tag{4.61b}$$

(c) Argue that the solutions for the algebraic expressions in Eqs. (4.51) can be expressed in the form

$$\tilde{x}_1(\omega) = \tilde{a}_1(\omega)\delta\big((\omega-\lambda_1)(\omega+\lambda_1)(\omega-\lambda_2)(\omega+\lambda_2)\big), \tag{4.62a}$$

$$\tilde{x}_2(\omega) = \tilde{a}_2(\omega)\delta\big((\omega-\lambda_1)(\omega+\lambda_1)(\omega-\lambda_2)(\omega+\lambda_2)\big), \tag{4.62b}$$

where $\tilde{a}_1(\omega)$ and $\tilde{a}_2(\omega)$ are arbitrary. Using the property of δ-functions show that

$$\frac{\tilde{x}_1(\omega)}{2\pi} = \frac{1}{8\pi\mu^2}\left[\frac{\tilde{a}_1(\lambda_1)}{\lambda_1}\delta(\omega-\lambda_1) + \frac{\tilde{a}_1(-\lambda_1)}{\lambda_1}\delta(\omega+\lambda_1) + \frac{\tilde{a}_1(\lambda_2)}{\lambda_2}\delta(\omega-\lambda_2) + \frac{\tilde{a}_1(-\lambda_2)}{\lambda_2}\delta(\omega+\lambda_2)\right], \tag{4.63a}$$

$$\frac{\tilde{x}_2(\omega)}{2\pi} = \frac{1}{8\pi\mu^2}\left[\frac{\tilde{a}_2(\lambda_1)}{\lambda_1}\delta(\omega-\lambda_1) + \frac{\tilde{a}_2(-\lambda_1)}{\lambda_1}\delta(\omega+\lambda_1) + \frac{\tilde{a}_2(\lambda_2)}{\lambda_2}\delta(\omega-\lambda_2) + \frac{\tilde{a}_2(-\lambda_2)}{\lambda_2}\delta(\omega+\lambda_2)\right]. \tag{4.63b}$$

The aribitrary coefficients are related due to the coupling in Eqs. (4.50). Thus, verify that

$$\tilde{a}_1(\pm\lambda_1) = \sqrt{\frac{\mu^2+\Delta^2}{\mu^2-\Delta^2}}\tilde{a}_2(\pm\lambda_1), \tag{4.64a}$$

$$\tilde{a}_1(\pm\lambda_2) = -\sqrt{\frac{\mu^2-\Delta^2}{\mu^2+\Delta^2}}\tilde{a}_2(\pm\lambda_2), \tag{4.64b}$$

and

$$\tilde{a}_2(\pm\lambda_1) = \sqrt{\frac{\mu^2-\Delta^2}{\mu^2+\Delta^2}}\tilde{a}_1(\pm\lambda_1), \tag{4.65a}$$

$$\tilde{a}_2(\pm\lambda_2) = -\sqrt{\frac{\mu^2+\Delta^2}{\mu^2-\Delta^2}}\tilde{a}_1(\pm\lambda_2). \tag{4.65b}$$

Using Eqs. (4.63) in the Fourier transform

$$x(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{-i\omega t}\tilde{x}_1(\omega) \tag{4.66}$$

and the redefinitions

$$A_1 = \frac{1}{8\pi\mu^2}\frac{\tilde{a}_1(\lambda_1)}{\lambda_1} \qquad B_1 = \frac{1}{8\pi\mu^2}\frac{\tilde{a}_1(-\lambda_1)}{\lambda_1}, \tag{4.67a}$$

$$A_2 = \frac{1}{8\pi\mu^2}\frac{\tilde{a}_1(\lambda_2)}{\lambda_2} \qquad B_2 = \frac{1}{8\pi\mu^2}\frac{\tilde{a}_1(-\lambda_2)}{\lambda_2}, \tag{4.67b}$$

which are determined by initial conditions, show that

$$x_1(t) = A_1 e^{-i\lambda_1 t} + B_1 e^{i\lambda_1 t} + A_2 e^{-i\lambda_2 t} + B_2 e^{i\lambda_2 t}, \tag{4.68a}$$

$$x_2(t) = \sqrt{\frac{\mu^2 - \Delta^2}{\mu^2 + \Delta^2}}\Big[A_1 e^{-i\lambda_1 t} + B_1 e^{i\lambda_1 t}\Big] - \sqrt{\frac{\mu^2 + \Delta^2}{\mu^2 - \Delta^2}}\Big[A_2 e^{-i\lambda_2 t} + B_2 e^{i\lambda_2 t}\Big]. \tag{4.68b}$$

(d) For initial conditions

$$x_1(0) = A, \quad \dot{x}_1(0) = 0, \quad x_2(0) = 0, \quad \dot{x}_2(0) = 0, \tag{4.69}$$

show that

$$x_1(t) = \frac{A}{2}\left[\left(1 + \frac{\Delta^2}{\mu^2}\right)\cos\lambda_1 t + \left(1 - \frac{\Delta^2}{\mu^2}\right)\cos\lambda_2 t\right], \tag{4.70a}$$

$$x_2(t) = \frac{A}{2}\frac{\omega_3^2}{\mu^2}\Big[\cos\lambda_1 t - \cos\lambda_2 t\Big]. \tag{4.70b}$$

(e) Sympathetic oscillations are characterized by the case

$$\Delta^2 \ll \omega_3^2 \tag{4.71}$$

when

$$\left(1 \pm \frac{\Delta^2}{\mu^2}\right) \sim 1, \quad \frac{\omega_3^2}{\mu^2} \sim 1, \quad \lambda_2^2 \sim \omega_2^2 + \omega_3^2, \quad \lambda_1^2 \sim \omega_1^2 - \omega_3^2, \tag{4.72}$$

and

$$x_1(t) = \frac{A}{2}\Big[\cos\lambda_1 t + \cos\lambda_2 t\Big] = A\cos\left(\frac{\lambda_1 - \lambda_2}{2}\right)t\cos\left(\frac{\lambda_1 + \lambda_2}{2}\right)t, \tag{4.73a}$$

$$x_2(t) = \frac{A}{2}\Big[\cos\lambda_1 t - \cos\lambda_2 t\Big] = A\sin\left(\frac{\lambda_1 - \lambda_2}{2}\right)t\cos\left(\frac{\lambda_1 + \lambda_2}{2}\right)t. \tag{4.73b}$$

Plot $x_1(t)$ anf $x_2(t)$ for $\omega_2 = 1.01\omega_1$ and $\omega_3 = 0.3\omega_1$, corresponding to $\omega_3 \sim 10\Delta$. The following 21 minute long Veritasium YouTube video discusses synchronization in a variety of systems,

https://youtu.be/t-_VPRCtiUg.

Discuss if these examples constitute sympathetic oscillations.

Chapter 5

Lagrangian multiplier

5.1 Equation of constraint

1. (**20 points.**) Consider the function describing a paraboloid

$$f(x,y) = a(x^2 + y^2). \tag{5.1}$$

A straight line on the xy plane, $y = mx + c$, can be interpreted as a condition of constraint

$$g(x,y) = y - mx - c = 0. \tag{5.2}$$

Let us determine the point on the line where the function $f(x,y)$ has an extremum value.

(a) Construct the function

$$F(x) = f(x, mx + c). \tag{5.3}$$

Using the extremum principle, $dF/dx = 0$, show that the point on the line where the function f is an extremum is

$$x = -\frac{mc}{1+m^2}, \quad y = \frac{c}{1+m^2}. \tag{5.4}$$

(b) Construct the function

$$h(x,y) = f(x,y) + \lambda g(x,y). \tag{5.5}$$

Evaluate $\boldsymbol{\nabla} h$, $\boldsymbol{\nabla} f$, and $\boldsymbol{\nabla} g$. Show that $\boldsymbol{\nabla} h = 0$ implies

$$x = \frac{\lambda m}{2a}, \quad y = -\frac{\lambda}{2a}. \tag{5.6}$$

Use this in the condition of constraint to derive

$$\lambda = -\frac{2ac}{1+m^2}. \tag{5.7}$$

Use the above expression for λ in Eq. (5.6) to find the point on the line where the function f is an extremum.

2. (**20 points.**) Spherical pendulum: Consider a pendulum that is suspended such that a mass m is able to move freely on the surface of a sphere of radius a (the length of the pendulum). The mass is then subject to the constraint

$$\phi = \frac{1}{2}(\mathbf{r}\cdot\mathbf{r} - a^2) = 0, \tag{5.8}$$

where a factor of 1/2 is introduced anticipating cancellations. Consider the Lagrangian function

$$L(\mathbf{r},\mathbf{v}) = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} + m\mathbf{g}\cdot\mathbf{r} + \mathbf{T}\cdot\boldsymbol{\nabla}\phi. \tag{5.9}$$

Here ϕ represents the equation for the surface of constraint, such that the gradient $\boldsymbol{\nabla}\phi$ is normal to the surface. The Lagrange multiplier $\mathbf{T}$ is interpreted as the force that is entrusted with the task of keeping the mass on the surface during motion. In this example of spherical pendulum $\mathbf{T}$ is the force of tension. My recording on the topic of planar pendulum, available at

https://youtu.be/dTU9p9VyeqE (45 minute video),

is a resource.

(a) Evaluate the gradient $\boldsymbol{\nabla}$ of the condition of constraint. Show that

$$\boldsymbol{\nabla}\phi = \mathbf{r}. \tag{5.10}$$

(Hint: Use $\boldsymbol{\nabla}\,\mathbf{r} = \mathbf{1}$.) Thus, show that

$$\mathbf{T}\cdot\boldsymbol{\nabla}\phi = \mathbf{T}\cdot\mathbf{r} \tag{5.11}$$

and

$$L(\mathbf{r},\mathbf{v}) = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} + m\mathbf{g}\cdot\mathbf{r} + \mathbf{T}\cdot\mathbf{r}. \tag{5.12}$$

(b) Using the Euler-Lagrange equations derive the equations of motion

$$m\mathbf{a} = m\mathbf{g} + \mathbf{T}, \tag{5.13}$$

where $\mathbf{a}$ is acceleration of mass m. Comparing Eq. (5.13) with the Newton equation of motion we recognize the Lagrangian multiplier to be the force of tension. In particular, this specifies the direction of $\mathbf{T}$ to be in the radially inward direction.

i. Equation of constraint: Find the projection of Newton's law of motion along the direction normal to the surface of constraint. Since $\hat{\mathbf{r}}$ is normal to the surface of the sphere we have

$$m\mathbf{a}\cdot\hat{\mathbf{r}} = m\mathbf{g}\cdot\hat{\mathbf{r}} + \mathbf{T}\cdot\hat{\mathbf{r}}, \tag{5.14}$$

which corresponds to

$$-m\dot{\phi}^2 a = mg\cos\phi + \mathbf{T}\cdot\hat{\mathbf{r}}. \tag{5.15}$$

ii. Equation of motion: By projecting in the tangential direction $\hat{\boldsymbol{\phi}}$ derive the equation of motion

$$a\ddot{\phi} = -g\sin\phi. \tag{5.16}$$

(c) Evaluate the canonical momentum

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v}. \tag{5.17}$$

(d) Construct the Hamiltonian using

$$H(\mathbf{r},\mathbf{p}) = \mathbf{v}\cdot\mathbf{p} - L(\mathbf{r},\mathbf{v}) \tag{5.18}$$

to be

$$H(\mathbf{r},\mathbf{p}) = \frac{p^2}{2m} - m\mathbf{g}\cdot\mathbf{r} - \mathbf{T}\cdot\mathbf{r}. \tag{5.19}$$

Derive the Hamilton equations of motion to be

$$\frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}}{m}, \tag{5.20a}$$

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{r}} = m\mathbf{g} + \mathbf{T}. \tag{5.20b}$$

Derive the statement of conservation of energy

$$\frac{dH}{dt} = 0 \tag{5.21}$$

starting from the Hamiltonian in Eq. (5.19) and using Hamilton equations of motion. You will also need to prove

$$\mathbf{r} \cdot \frac{d\mathbf{T}}{dt} = 0. \tag{5.22}$$

(e) Show that the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ satisfies the equation of motion

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times m\mathbf{g}, \tag{5.23}$$

and the angular momentum in the direction of $\mathbf{g}$ is conserved, that is,

$$\frac{d}{dt}(\mathbf{g} \cdot \mathbf{L}) = 0. \tag{5.24}$$

Area swept out by a particle as it moves along it's trajectory is given by

$$\frac{1}{2}\mathbf{r} \times d\mathbf{r}. \tag{5.25}$$

The rate at which this area changes is called the areal velocity. Thus, angular momentum is a measure of areal velocity. So, conclude the conservation of areal velocity in the direction of $\mathbf{g}$.

3. (**20 points.**) The Atwood machine consists of two masses m_1 and m_2 connected by a massless (inextensible) string passing over a massless pulley. See Figure 5.1. Massless pulley implies that tension in the string on both sides of the pulley is the same, say T. Further, the string being inextensible implies that the magnitude of the accelerations of both the masses are the same. Let $m_2 > m_1$.

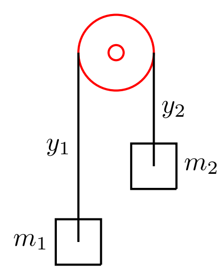

Figure 5.1: example

(a) Let lengths y_1 and y_2 be positive distances from the pulley to the masses such that the accelerations $a_1 = \ddot{y}_1$ and $a_2 = \ddot{y}_2$ satisfy $a_2 = -a_1 = a$. Using Newton's law determine the equations of motion for the masses to be

$$m_2 g - T = m_2 a, \tag{5.26a}$$
$$m_1 g - T = -m_1 a. \tag{5.26b}$$

Thus, show that

$$\text{Equation of motion:} \qquad a = \left(\frac{m_2 - m_1}{m_2 + m_1}\right) g, \tag{5.27a}$$

$$\text{Equation of constraint:} \qquad T = \frac{2m_1 m_2 g}{(m_1 + m_2)}. \tag{5.27b}$$

(b) The constraint among the dynamical variables y_1 and y_2 is

$$y_1 + y_2 = L, \tag{5.28}$$

where L is the total length of the string connecting the two masses. Show that the Lagrangian for Atwood's machine can be expressed in terms of a single dynamical variable, say y_2, as

$$L(y_2, \dot{y}_2) = \frac{1}{2}(m_1 + m_2)\dot{y}_2^2 + (m_2 - m_1)g y_2. \tag{5.29}$$

Find the corresponding Euler-Lagrange equation.

(c) Using the idea of Lagrange multiplier construct another Lagrangian

$$L(y_1, y_2, \dot{y}_1, \dot{y}_2) = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 + m_1 g y_1 + m_2 g y_2 - T\frac{\partial}{\partial y_1}(y_1 + y_2 - L)^2\frac{1}{2}, \tag{5.30}$$

where T here is interpreted as the Lagrangian multiplier. Find the corresponding Euler-Lagrange equations.

4. (**20 points.**) A mass m_2 is connected to another mass m_1 by a massless (inextensible) string passing over a massless pulley, as described in Figure 5.2. Surfaces are frictionless. Massless pulley implies that tension in the string on both sides of the pulley is the same, say T. Further, the string being inextensible implies that the magnitude of the accelerations of both the masses are the same.

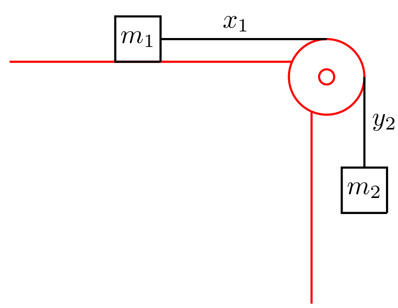

Figure 5.2: Problem 4

(a) Let lengths x_1 and y_2 be positive distances from the pulley to the masses such that the accelerations $a_1 = \ddot{x}_1$ and $a_2 = \ddot{y}_2$ satisfy $a_2 = -a_1 = a$. Using Newton's law determine the equations of motion for the masses to be

$$m_2 g - T = m_2 a, \tag{5.31a}$$
$$T = m_1 a. \tag{5.31b}$$

Thus, show that

$$\text{Equation of motion:} \qquad a = \left(\frac{m_2}{m_1 + m_2}\right) g, \tag{5.32a}$$

$$\text{Equation of constraint:} \qquad T = \left(\frac{m_2}{m_1 + m_2}\right) m_1 g. \tag{5.32b}$$

(b) The constraint among the dynamical variables x_1 and y_2 is

$$x_1 + y_2 = L, \tag{5.33}$$

where L is the total length of the string connecting the two masses. Show that the Lagrangian describing the motion can be expressed in terms of a single dynamical variable, say y_2, as

$$L(y_2, \dot{y}_2) = \frac{1}{2}(m_1 + m_2)\dot{y}_2^2 + m_2 g y_2. \tag{5.34}$$

Find the corresponding Euler-Lagrange equation.

(c) Using the idea of Lagrange multiplier construct another Lagrangian

$$L(x_1, y_2, \dot{x}_1, \dot{y}_2) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 + m_2 g y_2 - T\frac{\partial}{\partial x_1}(x_1 + y_2 - L)^2\frac{1}{2}, \tag{5.35}$$

where T here is interpreted as the Lagrangian multiplier. Find the corresponding Euler-Lagrange equations.

5. (**20 points.**) (**Incomplete, Needs refinement.**) A mass m slides down a frictionless ramp that is inclined at an angle α with respect to the horizontal. Assume uniform acceleration due to gravity g in the vertical downward direction. In terms of a suitable dynamical variable write a Lagrangian that describes the motion of the mass.

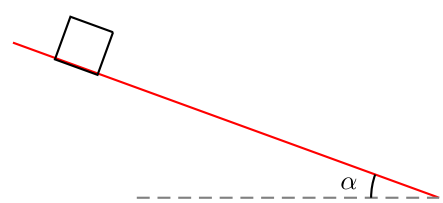

Figure 5.3: Problem 5.

(a) Relevant coordinates are related by a rotation

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5.36}$$

(b) In terms of dynamical variables $\bar{x}$ and $\bar{y}$ the constraint

$$y = y_0 \tag{5.37}$$

is given by

$$\bar{x}\sin\alpha + \bar{y}\cos\alpha = y_0, \tag{5.38}$$

such that $\ddot{\bar{x}}\sin\alpha = -\ddot{\bar{y}}\cos\alpha$. Show that the Lagrangian is, say in terms of $\bar{y}$,

$$L(\bar{y}, \dot{\bar{y}}) = \frac{1}{2}m\dot{\bar{x}}^2 + \frac{1}{2}m\dot{\bar{y}}^2 - mg\bar{y} \tag{5.39a}$$

$$= \frac{1}{2}m\dot{\bar{y}}^2\csc^2\alpha - mg\bar{y}. \tag{5.39b}$$

Show that the equations of motion are

$$\ddot{\bar{y}} = -g\sin^2\alpha. \tag{5.40}$$

In conjunction with the constraint show that this implies

$$\sqrt{\ddot{\bar{x}} + \ddot{\bar{y}}} = g\sin\alpha. \tag{5.41}$$

(c) In terms of dynamical variables x and y we have $\ddot{y} = 0$. Show that the Lagrangian is, say in terms of x,

$$L(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 + mgx \sin\alpha. \tag{5.42}$$

Find the corresponding Euler-Lagrange equation.

(d) Using the idea of Lagrange multiplier construct another Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{y}^2 - mg(\sin\alpha x + \cos\alpha y) + N \frac{\partial}{\partial y}(y - y_0)^2 \frac{1}{2}, \tag{5.43}$$

where N here is interpreted as the Lagrangian multiplier. Find the corresponding Euler-Lagrange equations.

6. (**20 points.**) Consider a wheel rolling on a horizontal surface. The following distinct types of motion are

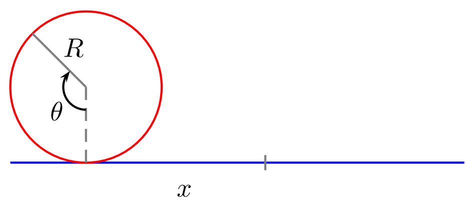

Figure 5.4: Problem 6.

possible for the wheel:

$$\begin{aligned} x &< \theta R, && \text{slipping (e.g. in snow)}, \\ x &= \theta R, && \text{perfect rolling}, \\ x &> \theta R, && \text{sliding (e.g. on ice)}. \end{aligned} \tag{5.44}$$

Differentiation of the these relations leads to the characterizations, $v < \omega R$, $v = \omega R$, and $v > \omega R$, respectively, where $v = \dot{x}$ is the linear velocity and $\omega = \dot{\theta}$ is the angular velocity. Assuming the wheel is perfectly rolling, at a given instant of time, the tendency of motion could be to slip, to keep on perfectly rolling, or to slide.

Deduce that while perfectly rolling the relative motion of the point on the wheel that is in contact with the surface with respect to the surface is exactly zero. Thus, conclude that the force of friction on the wheel is zero. The analogy is a mass at rest on a horizontal surface. However, while perfectly rolling, it is possible to have the tendency to slip or slide without actually slipping of sliding. The analogy is that of a mass at rest under the action of an external force and the force of friction. In these cases the force of friction is that of static friction and it acts in the forward or backward direction.

In the following we differentiate between the following:

(a) Tendency of the wheel is to slip (without actually slipping) while perfectly rolling.

(b) Tendency of the wheel is to keep on perfectly rolling.

(c) Tendency of the wheel is to slide (without actually sliding) while perfectly rolling.

Deduce the direction of the force of friction in the above cases. Determine if the friction is working against linear acceleration or angular acceleration.

Perfect rolling involves the contraint $x = \theta R$. Thus, using the D'Alembert's principle and idea of Lagrange multiplier we can write the Lagrangian for a perfectly rolling wheel on a horizontal surface to be

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I\dot{\theta}^2 - F_s(x - \theta R), \tag{5.45}$$

where m is the mass of the wheel, I is the moment of inertia of the wheel, and F_s is the Lagrangian multiplier. Using D'Alembert's principle give an interpretation for the Lagrangian multiplier F_s. What is the dimension of F_s? Infer the sign of F_s for the cases when the tendency of the wheel is to slip or slide while perfectly rolling.

7. (**20 points.**) Consider two discs of radii r_1 and r_2, and moment of inertia I_1 and I_2. Disc 1 is free to roll about an axis parallel to z axis passing through its center O_1. Similarly, disc 2 is free to roll about an axis parallel to z axis passing through its center O_2. Further, the center of disc 2 is free to move on a circle of radii $(r_1 + r_2)$. Let I_3 be the moment of inertia of disc 2 about the axis passing through O_1. See Figure 7. Assume gravity in the direction of z axis and no motion in the z direction so that gravity effects are irrelevant. The two discs are in contact with sufficient friction between them such that the resultant motion leads to perfect rolling of the surfaces,

$$\theta_1 r_1 = \theta_2 r_2. \tag{5.46}$$

Here θ_1 and θ_2 are angular displacements of the respective discs about the axes O_1 and O_2. Further, the angular displacement of the axis O_2 about the axis O_1 is parametrized by the angular displacement α_2. Assume the discs are rolling under the action of no external torques.

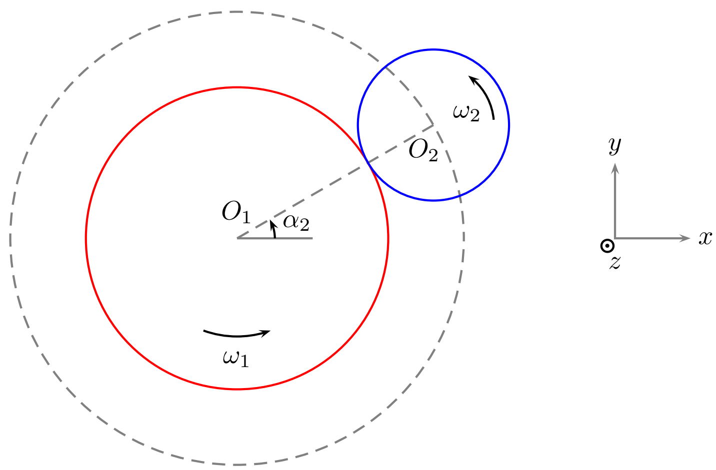

Figure 5.5: Problem 7.

(a) Show that the Lagrangian for this system in terms of the coordinates θ_1 and α_2, and their derivatives, is

$$L(\theta_1, \dot{\theta}_1, \alpha_2, \dot{\alpha}_2) = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} I_3 \dot{\alpha}_2^2 \tag{5.47a}$$

$$= \frac{1}{2}\left(I_1 + I_2 \frac{r_1^2}{r_2^2}\right) \dot{\theta}_1^2 + \frac{1}{2} I_3 \dot{\alpha}_2^2, \tag{5.47b}$$

where the equation of contraint has been used to replace θ_2. Determine the equations of motion to be

$$\left(I_1 + I_2 \frac{r_1^2}{r_2^2}\right) \ddot{\theta}_1 = 0 \tag{5.48}$$

and

$$I_3 \ddot{\alpha}_2 = 0. \tag{5.49}$$

These imply $\ddot{\theta}_1 = 0$ and $\ddot{\alpha}_2 = 0$ in the absence of external torque.

(b) Show that the Lagrangian for this system in terms of the coordinates θ_1, θ_2, and α_2, and their derivatives, is

$$L(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, \alpha_2, \dot{\alpha}_2) = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} I_3 \dot{\alpha}^2 + \lambda(\theta_1 r_1 - \theta_2 r_2), \tag{5.50}$$

where the contraint has been introduced with Lagrange multiplier λ. Determine the equations of motion to be

$$I_1 \ddot{\theta}_1 = \lambda r_1, \tag{5.51a}$$
$$I_2 \ddot{\theta}_2 = -\lambda r_2, \tag{5.51b}$$
$$I_3 \ddot{\alpha}_2 = 0. \tag{5.51c}$$

Combine Eqs.(5.51a) and (5.51b) and show that it is consistent with Eq.(5.48).

(c) Which quantity relates to the Lagrange multiplier λ.

(d) In the absence of external torque and $\dot{\alpha}_2 = 0$ initially deduce that the center of mass of disc 2 is stationary.

Chapter 6

Stationary action principle (with variations at boundary)

6.1 Variation at the boundary

1. (**20 points.**) The Hamiltonian is defined by the relation

$$H(p_i, q_i, t) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i, t). \tag{6.1}$$

Show that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{6.2}$$

Under what circumstances is H interpreted as the energy of the system?

2. (**30 points.**) Consider the Lagrangian

$$L = \frac{1}{2} m \left(\frac{d\mathbf{r}}{dt} \right)^2 - V(\mathbf{r}, t). \tag{6.3}$$

(a) Show that principle of stationary action with respect to $\delta\mathbf{r}$ implies Newton's second law

$$m \frac{d^2\mathbf{r}}{dt^2} = -\boldsymbol{\nabla} V. \tag{6.4}$$

(b) Show that principle of stationary action with respect to δt implies

$$\frac{d}{dt} \left[\frac{1}{2} m \left(\frac{d\mathbf{r}}{dt} \right)^2 + V \right] = \frac{\partial V}{\partial t}, \tag{6.5}$$

which for a static potential, $\partial V/\partial t = 0$, is the statement of conservation of energy.

(c) Show that the invariance of the total time derivative term, that gets contributions only from the end points, under an infinitesimal rigid rotation

$$\mathbf{r}' = \mathbf{r} - \delta\mathbf{r}, \qquad \delta\mathbf{r} = \delta\boldsymbol{\omega} \times \mathbf{r}, \tag{6.6}$$

implies the conservation of total angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

3. (**40 points.**) In terms of the Lagrangian function $L(\mathbf{r}, \mathbf{v}, t)$ the action functional $W[\mathbf{r}; t_1, t_2]$ is defined as

$$W[\mathbf{r}; t_1, t_2] = \int_{t_1}^{t_2} dt \, L(\mathbf{r}, \mathbf{v}, t), \tag{6.7}$$

where $\mathbf{v} = d\mathbf{r}/dt$.

(a) For arbitrary infinitesimal variations in the path

$$\bar{\mathbf{r}}(t) = \mathbf{r}(t) - \delta\mathbf{r}(t), \tag{6.8}$$

and infinitesimal general time transformation

$$\bar{t} = t - \delta t(t), \tag{6.9}$$

the change in action is given by

$$\begin{aligned} \delta W &= \int_{t_1}^{t_2} dt \frac{d}{dt}\Big[\mathbf{p}\cdot\delta\mathbf{r} - H\delta t\Big] \\ &\quad + \int_{t_1}^{t_2} dt \left[\delta t \left(\frac{dH}{dt} + \frac{\partial L}{\partial t}\right) + \delta\mathbf{r}\cdot\left(\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}}\right)\right], \end{aligned} \tag{6.10}$$

where the canonical momentum and the Hamiltonian are defined as

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} \quad \text{and} \quad H = \mathbf{v}\cdot\mathbf{p} - L \tag{6.11}$$

respectively.

(b) The change in the action due to variations in path is captured in the functional derivative

$$\frac{\delta W}{\delta \mathbf{r}(t)} = \left(\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}}\right) + \Big[\delta(t-t_2) - \delta(t-t_1)\Big]\mathbf{p}. \tag{6.12}$$

The change in the action due to time transformation is captured in the functional derivative

$$\frac{\delta W}{\delta t(t)} = \left(\frac{dH}{dt} + \frac{\partial L}{\partial t}\right) - \Big[\delta(t-t_2) - \delta(t-t_1)\Big]H. \tag{6.13}$$

(c) In terms of the Hamiltonian the action takes the form

$$W[\mathbf{r},\mathbf{p};t_1,t_2] = \int_{t_1}^{t_2} dt\Big[\mathbf{v}\cdot\mathbf{p} - H(\mathbf{r},\mathbf{p},t)\Big]. \tag{6.14}$$

(d) Show that for for arbitrary infinitesimal variations in coordinate and momentum

$$\bar{\mathbf{r}}(t) = \mathbf{r}(t) - \delta\mathbf{r}(t) \quad \text{and} \quad \bar{\mathbf{p}}(t) = \mathbf{p}(t) - \delta\mathbf{p}(t), \tag{6.15}$$

and infinitesimal general time transformation, the change in action is given by

$$\begin{aligned} \delta W &= \int_{t_1}^{t_2} dt \frac{d}{dt}\Big[\mathbf{p}\cdot\delta\mathbf{r} - H\delta t\Big] \\ &\quad + \int_{t_1}^{t_2} dt \left[\delta t \left(\frac{dH}{dt} - \frac{\partial H}{\partial t}\right) - \delta\mathbf{r}\cdot\left(\frac{d\mathbf{p}}{dt} + \frac{\partial H}{\partial \mathbf{r}}\right) + \delta\mathbf{p}\cdot\left(\frac{d\mathbf{r}}{dt} - \frac{\partial H}{\partial \mathbf{p}}\right)\right]. \end{aligned} \tag{6.16}$$

4. (**20 points.**) In terms of the Lagrangian function $L(\mathbf{r},\mathbf{v},t)$ the action $W[\mathbf{r};t_1,t_2]$ is defined as

$$W[\mathbf{r};t_1,t_2] = \int_{t_1}^{t_2} dt\, L(\mathbf{r},\mathbf{v},t), \tag{6.17}$$

where $\mathbf{v} = d\mathbf{r}/dt$. Find the change in the action under an infinitesimal general time transformation

$$\bar{t} = t - \delta t(t), \quad \delta t(t_1) = 0, \quad \delta t(t_1) = 0. \tag{6.18}$$

In paticular, evaluate the functional derivative

$$\frac{\delta W}{\delta t(t)} \tag{6.19}$$

for the variation $\delta t(t)$ satisfying the constraints of Eq. (6.18).

6.2 Symmetry and conservation principles

5. (**20 points.**) Consider infinitesimal rigid translation in space, described by

$$\delta \mathbf{r} = \delta \boldsymbol{\epsilon}, \quad \delta \mathbf{p} = 0, \quad \delta t = 0, \tag{6.20}$$

where $\delta \boldsymbol{\epsilon}$ is independent of position and time.

(a) Show that the change in the action due to the above translation is

$$\frac{\delta W}{\delta \boldsymbol{\epsilon}} = -\int_{t_1}^{t_2} dt \frac{\partial H}{\partial \mathbf{r}}. \tag{6.21}$$

(b) Show, separately, that the change in the action under the above translation is also given by

$$\frac{\delta W}{\delta \boldsymbol{\epsilon}} = \int_{t_1}^{t_2} dt \frac{d\mathbf{p}}{dt} = \mathbf{p}(t_2) - \mathbf{p}(t_1). \tag{6.22}$$

This states that linear momentum is the generator of rigid translation in space.

(c) Together, we have the relation connecting linear momentum and impluse,

$$\mathbf{p}(t_2) - \mathbf{p}(t_1) = \int_{t_1}^{t_2} dt\, \mathbf{F}, \tag{6.23}$$

where we used

$$\mathbf{F} = -\frac{\partial H}{\partial \mathbf{r}}. \tag{6.24}$$

The system is defined to have translational symmetry when the action does not change under rigid translation. Show that a system has translation symmetry when

$$-\frac{\partial H}{\partial \mathbf{r}} = 0. \tag{6.25}$$

That is, when the Hamiltonian is independent of position. Or, when the force $\mathbf{F} = -\partial H / \partial \mathbf{r} = 0$.

(d) Deduce that the linear momentum is conserved, that is,

$$\mathbf{p}(t_1) = \mathbf{p}(t_2), \tag{6.26}$$

when the action has translation symmetry.

6. (**20 points.**) Consider infinitesimal rigid translation in time, described by

$$\delta \mathbf{r} = 0, \quad \delta \mathbf{p} = 0, \quad \delta t = \delta \epsilon, \tag{6.27}$$

where $\delta \epsilon$ is independent of position and time.

(a) Show that the change in the action due to the above translation is

$$\frac{\delta W}{\delta \epsilon} = -\int_{t_1}^{t_2} dt \frac{\partial H}{\partial t}. \tag{6.28}$$

(b) Show, separately, that the change in the action under the above translation is also given by

$$\frac{\delta W}{\delta \epsilon} = -\int_{t_1}^{t_2} dt \frac{dH}{dt} = -H(t_2) + H(t_1). \tag{6.29}$$

This states that Hamiltonian is the generator of rigid translation in time.

(c) The system is defined to have translational symmetry when the action does not change under rigid translation. Show that a system has translation symmetry when

$$-\frac{\partial H}{\partial t} = 0. \tag{6.30}$$

That is, when the Hamiltonian is independent of time.

(d) Deduce that the Hamiltonian is conserved, that is,

$$H(t_1) = H(t_2), \tag{6.31}$$

when the action has translation symmetry.

7. (**20 points.**) A general rotation in 3-dimensions can be written in terms of consecutive rotations about x, y, and z axes,

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{6.32}$$

For infinitesimal rotations we use

$$\cos\theta_i \sim 1, \tag{6.33a}$$

$$\sin\theta_i \sim \theta_i \to \delta\theta_i, \tag{6.33b}$$

to obtain

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & \delta\theta_3 & -\delta\theta_2 \\ -\delta\theta_3 & 1 & \delta\theta_1 \\ \delta\theta_2 & -\delta\theta_1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{6.34}$$

Show that this corresponds to the vector relation

$$\mathbf{r}' = \mathbf{r} - \delta\boldsymbol{\theta} \times \mathbf{r} \tag{6.35}$$

such that

$$\delta\mathbf{r} = \delta\boldsymbol{\theta} \times \mathbf{r}, \tag{6.36}$$

where

$$\mathbf{r} = x_1\hat{\mathbf{x}} + x_2\hat{\mathbf{y}} + x_3\hat{\mathbf{z}}, \tag{6.37a}$$

$$\delta\boldsymbol{\theta} = \delta\theta_1\hat{\mathbf{x}} + \delta\theta_2\hat{\mathbf{y}} + \delta\theta_3\hat{\mathbf{z}}. \tag{6.37b}$$

As a particular example, verify that a rotation about the direction $\hat{\mathbf{z}}$ by an infinitesimal (azimuth) angle $\delta\phi$ is described by

$$\delta\boldsymbol{\theta} = \hat{\mathbf{z}}\,\delta\phi. \tag{6.38}$$

The corresponding infinitesimal transformation in $\mathbf{r}$ is given by

$$\delta\mathbf{r} = \delta\phi\,\hat{\mathbf{z}} \times \mathbf{r} = \hat{\boldsymbol{\phi}}\,\rho\delta\phi, \tag{6.39}$$

where ρ and ϕ are the cylindrical coordinates defined as

$$\hat{\mathbf{z}} \times \mathbf{r} = \boldsymbol{\phi} \quad \text{and} \quad |\hat{\mathbf{z}} \times \mathbf{r}| = \rho. \tag{6.40}$$

Observe that, in rectangular coordinates $\rho\hat{\boldsymbol{\phi}} = x\hat{\mathbf{y}} - y\hat{\mathbf{x}}$.

8. (**20 points.**) Consider infinitesimal rigid rotation, described by

$$\delta\mathbf{r} = \delta\boldsymbol{\theta} \times \mathbf{r}, \quad \delta\mathbf{p} = \delta\boldsymbol{\theta} \times \mathbf{p}, \quad \delta t = 0, \tag{6.41}$$

where $d\delta\boldsymbol{\theta}/dt = 0$.

(a) Show that the variation in the action under the above rotation is

$$\frac{\delta W}{\delta \boldsymbol{\theta}} = \int_{t_1}^{t_2} dt \left[\mathbf{r} \times \frac{\partial L}{\partial \mathbf{r}} + \mathbf{p} \times \frac{\partial L}{\partial \mathbf{p}} \right] \tag{6.42}$$

or

$$\frac{\delta W}{\delta \boldsymbol{\theta}} = -\int_{t_1}^{t_2} dt \left[\mathbf{r} \times \frac{\partial H}{\partial \mathbf{r}} + \mathbf{p} \times \frac{\partial H}{\partial \mathbf{p}} \right]. \tag{6.43}$$

(b) Show, separately, that the change in the action under the above rotation is also given by

$$\frac{\delta W}{\delta \boldsymbol{\theta}} = \int_{t_1}^{t_2} dt \frac{d\mathbf{L}}{dt} = \mathbf{L}(t_2) - \mathbf{L}(t_1), \tag{6.44}$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum. This states that angular momentum momentum is the generator of rigid rotation in space.

(c) The system is defined to have rotational symmetry when the action does not change under rigid rotation. Show that a system has rotation symmetry when

$$\mathbf{r} \times \frac{\partial L}{\partial \mathbf{r}} = 0 \quad \text{and} \quad \mathbf{p} \times \frac{\partial L}{\partial \mathbf{p}} = 0, \tag{6.45}$$

or

$$\mathbf{r} \times \frac{\partial H}{\partial \mathbf{r}} = 0 \quad \text{and} \quad \mathbf{p} \times \frac{\partial H}{\partial \mathbf{p}} = 0. \tag{6.46}$$

Show that this corresponds to

$$\frac{\partial L}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \phi} = 0, \tag{6.47}$$

or

$$\frac{\partial H}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial H}{\partial \phi} = 0. \tag{6.48}$$

That is, when the Lagrangian is independent of angular coordinates θ and ϕ.

(d) Deduce that the anglular momentum is conserved, that is,

$$\mathbf{L}(t_1) = \mathbf{L}(t_2), \tag{6.49}$$

when the action has rotational symmetry.

9. (**20 points.**) Noether's theorem, in the context of rotational symmetry, states that if the Lagrangian does not change under an infinitesimal rigid rotation, then the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved. Prove that the converse of Noether's theorem is also true. For simplicity consider velocity independent potentials.

Chapter 7

Canonical transformation

7.1 Hamilton-Jacobi equation

1. (**40 points.**) The Hamiltonian for the motion of a particle of mass m in a constant gravitational field $\mathbf{g} = -g\hat{\mathbf{z}}$ is

$$H(z,p,t) = \frac{p^2}{2m} + mgz. \tag{7.1}$$

(a) Show that the Hamilton equations of motion are

$$\frac{dz}{dt} = \frac{p}{m}, \tag{7.2a}$$

$$\frac{dp}{dt} = -mg. \tag{7.2b}$$

(b) Show that the Hamilton-Jacobi equation

$$-\frac{\partial W}{\partial t} = H\left(z, \frac{\partial W}{\partial z}, t\right), \tag{7.3}$$

in terms of Hamilton's principal function $W(z,t)$ is given by

$$-\frac{\partial W}{\partial t} = \frac{1}{2m}\left(\frac{\partial W}{\partial z}\right)^2 + mgz. \tag{7.4}$$

Further, show that

$$W(z,t) = -Et - \frac{2}{3}\frac{\sqrt{2m}}{mg}(E - mgz)^{\frac{3}{2}} \tag{7.5}$$

is a solution to the Hamilton-Jacobi equation up to a constant.

(c) Hamilton's principal function allows us to identify canonical transformations $Q = Q(z,p,t)$ and $P = P(z,p,t)$, such that

$$\frac{\partial W}{\partial z} = p, \qquad \frac{\partial W}{\partial Q} = -P, \qquad \frac{\partial W}{\partial t} = -H, \tag{7.6a}$$

$$\frac{\partial W}{\partial p} = 0, \qquad \frac{\partial W}{\partial P} = 0, \tag{7.6b}$$

with the feature that the new coordinates are constants of motion,

$$\frac{dQ}{dt} = 0 \qquad \text{and} \qquad \frac{dP}{dt} = 0. \tag{7.7}$$

To this end, choose $Q = E$ and then evaluate

$$P = -\frac{\partial W}{\partial Q} = t + \frac{p}{mg}. \tag{7.8}$$

Hint: Use $p = \frac{\partial W}{\partial z}$.

(d) Show that

$$\begin{aligned} Q &= \frac{p^2}{2m} + mgz, && (7.9a) \\ P &= t + \frac{p}{mg}, && (7.9b) \end{aligned}$$

is a canonical transformation. That is, show that $[Q,P]^{\text{P.B.}}_{z,p} = 1$. Further, verify that

$$\begin{aligned} \frac{dQ}{dt} &= 0, && (7.10a) \\ \frac{dP}{dt} &= 0, && (7.10b) \\ K(Q,P,t) &= H(z,p,t) + \frac{\partial W}{\partial t} = 0. && (7.10c) \end{aligned}$$

7.2 Poisson braket

1. (**40 points.**) Type notes dated 2022 Mar 29.

2. (**20 points.**) Given F and G are constants of motion, that is

$$\big[F,H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0 \qquad \text{and} \qquad \big[G,H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{7.11}$$

Then, using Jacobi's identity, show that $\big[F,G\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}$ is also a constant of motion. Thus, conclude the following:

(a) If L_x and L_y are constants of motion, then L_z is also a constant of motion.

(b) If p_x and L_z are constants of motion, then p_y is also a constant of motion.

3. (**20 points.**) Hamiltonian for the motion of a ball (along the radial direction) near the surface of Earth is given by

$$H(z,p_z) = \frac{p_z^2}{2m} - mgz. \tag{7.12}$$

(a) Determine the equations of motions using

$$\frac{dz}{dt} = \frac{\partial H}{\partial p_z} \qquad \text{and} \qquad \frac{dp_z}{dt} = -\frac{\partial H}{\partial z}. \tag{7.13}$$

Then, solve the coupled differential equations to find the familiar elementary solution

$$z(t) = z(0) + \frac{p_z(0)}{m}t + \frac{1}{2}gt^2 \tag{7.14a}$$

and

$$p_z(t) = p_z(0) + mg\,t. \tag{7.14b}$$

(b) Next, determine the equations of motion using

$$\frac{dz(t)}{dt} = \big[z(t), H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \tag{7.15a}$$
$$\frac{dp_z(t)}{dt} = \big[p_z(t), H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}. \tag{7.15b}$$

Evaluate

$$\frac{d^2z(t)}{dt^2} = \Big[\big[z(t), H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, H\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, \tag{7.16a}$$
$$\frac{d^3z(t)}{dt^3} = \Big[\Big[\big[z(t), H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, H\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, H\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, \tag{7.16b}$$
$$\vdots$$

and

$$\frac{d^2p_z(t)}{dt^2} = \Big[\big[p_z(t), H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, H\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, \tag{7.17a}$$
$$\frac{d^3p_z(t)}{dt^3} = \Big[\Big[\big[p_z(t), H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, H\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, H\Big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p},0}, \tag{7.17b}$$
$$\vdots$$

Then, using

$$z(t) = z(0) + \frac{t}{1!}\left\{\frac{dz}{dt}\right\}_{t=0} + \frac{t^2}{2!}\left\{\frac{d^2z}{dt^2}\right\}_{t=0} + \cdots \tag{7.18}$$

and

$$p_z(t) = p_z(0) + \frac{t}{1!}\left\{\frac{dp_z}{dt}\right\}_{t=0} + \frac{t^2}{2!}\left\{\frac{d^2p_z}{dt^2}\right\}_{t=0} + \cdots \tag{7.19}$$

rederive the solutions in Eqs. (7.14).

4. (**20 points.**) Harmonic oscillations are described by the Hamiltonian

$$H(x,p) = \frac{1}{2}p^2 + \frac{1}{2}x^2. \tag{7.20}$$

(a) Determine the equations of motions using

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \qquad \text{and} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}. \tag{7.21}$$

Then, solve the coupled differential equations to find the solutions

$$x(t) = x(0)\cos t + p(0)\sin t, \tag{7.22a}$$
$$p(t) = -x(0)\sin t + p(0)\cos t, \tag{7.22b}$$

where $x(0)$ and $p(0)$ are given using the intial conditions at $t = 0$.

(b) Next, determine the equations of motion using

$$\frac{dx(t)}{dt} = \big[x(t), H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \tag{7.23a}$$
$$\frac{dp(t)}{dt} = \big[p(t), H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}. \tag{7.23b}$$

Evaluate the following nested commutation relations,

$$\big[\dots\big[\big[x(t),H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}},H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}},\dots\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} \tag{7.24}$$

and

$$\big[\dots\big[\big[p(t),H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}},H\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}},\dots\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \tag{7.25}$$

iteratively. Thus, evaluate

$$\frac{d^n x(t)}{dt^n} = \begin{cases} x(t)\, i^n, & \text{for} \quad n = 0,2,4,\dots, \\ p(t)\, i^{n-1}, & \text{for} \quad n = 1,3,5,\dots, \end{cases} \tag{7.26}$$

and

$$\frac{d^n p(t)}{dt^n} = \begin{cases} p(t)\, i^n, & \text{for} \quad n = 0,2,4,\dots, \\ -x(t)\, i^{n-1}, & \text{for} \quad n = 1,3,5,\dots. \end{cases} \tag{7.27}$$

Then, using the above in the Taylor expansion,

$$x(t) = x(0) + \frac{t}{1!}\left\{\frac{dx}{dt}\right\}_{t=0} + \frac{t^2}{2!}\left\{\frac{d^2x}{dt^2}\right\}_{t=0} + \cdots \tag{7.28}$$

and

$$p(t) = p(0) + \frac{t}{1!}\left\{\frac{dp}{dt}\right\}_{t=0} + \frac{t^2}{2!}\left\{\frac{d^2p}{dt^2}\right\}_{t=0} + \cdots, \tag{7.29}$$

rederive the solutions in Eqs. (7.22).

7.2.1 Lie Algebra of Poisson braket

1. (**40 points.**) For two functions

$$A = A(\mathbf{x},\mathbf{p},t), \tag{7.30a}$$
$$B = B(\mathbf{x},\mathbf{p},t), \tag{7.30b}$$

the Poisson braket with respect to the canonical variables $\mathbf{x}$ and $\mathbf{p}$ is defined as

$$\big[A,B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} \equiv \frac{\partial A}{\partial \mathbf{x}}\cdot\frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}}\cdot\frac{\partial B}{\partial \mathbf{x}}. \tag{7.31}$$

Show that the Poisson braket satisfies the conditions for a Lie algebra. That is, show that

(a) Antisymmetry:

$$\big[A,B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = -\big[B,A\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}. \tag{7.32}$$

(b) Bilinearity: (a and b are numbers.)

$$\big[aA+bB,C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = a\big[A,C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + b\big[B,C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}. \tag{7.33}$$

Further show that

$$\big[AB,C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = A\big[B,C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[A,C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B. \tag{7.34}$$

(c) Jacobi's identity:

$$\big[A,\big[B,C\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[B,\big[C,A\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[C,\big[A,B\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{7.35}$$

2. (**40 points.**) Show that the commutator of two matrices,
$$[\mathbf{A},\mathbf{B}] \equiv \mathbf{AB} - \mathbf{BA}, \tag{7.36}$$
satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that
   (a) Antisymmetry:
   $$[\mathbf{A},\mathbf{B}] = -[\mathbf{B},\mathbf{A}]. \tag{7.37}$$
   (b) Bilinearity: (a and b are numbers.)
   $$[a\mathbf{A}+b\mathbf{B},\mathbf{C}] = a[\mathbf{A},\mathbf{C}] + b[\mathbf{B},\mathbf{C}]. \tag{7.38}$$
   Further show that
   $$[\mathbf{AB},\mathbf{C}] = \mathbf{A}[\mathbf{B},\mathbf{C}] + [\mathbf{A},\mathbf{C}]\mathbf{B}. \tag{7.39}$$
   (c) Jacobi's identity:
   $$[\mathbf{A},[\mathbf{B},\mathbf{C}]] + [\mathbf{B},[\mathbf{C},\mathbf{A}]] + [\mathbf{C},[\mathbf{A},\mathbf{B}]] = 0. \tag{7.40}$$
3. (**40 points.**) Show that the vector product of two vectors, in this problem denoted using
$$[\mathbf{A},\mathbf{B}]_v \equiv \mathbf{A}\times\mathbf{B}, \tag{7.41}$$
satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that
   (a) Antisymmetry:
   $$[\mathbf{A},\mathbf{B}]_v = -[\mathbf{B},\mathbf{A}]_v. \tag{7.42}$$
   (b) Bilinearity: (a and b are numbers.)
   $$[a\mathbf{A}+b\mathbf{B},\mathbf{C}]_v = a[\mathbf{A},\mathbf{C}]_v + b[\mathbf{B},\mathbf{C}]_v. \tag{7.43}$$
   Further show that
   $$[\mathbf{A}\times\mathbf{B},\mathbf{C}]_v = \mathbf{A}\times[\mathbf{B},\mathbf{C}]_v + [\mathbf{A},\mathbf{C}]_v\times\mathbf{B}. \tag{7.44}$$
   (c) Jacobi's identity:
   $$[\mathbf{A},[\mathbf{B},\mathbf{C}]_v]_v + [\mathbf{B},[\mathbf{C},\mathbf{A}]_v]_v + [\mathbf{C},[\mathbf{A},\mathbf{B}]_v]_v = 0. \tag{7.45}$$
4. (**40 points.**) Construct a problem on Heisenberg group, Weyl algebra, Bergman-Segal space.
5. (**40 points.**) (Refer Sec. 21 Dirac's QM book.)
The product rule for Poisson braket can be stated in the following different forms:
$$[A_1A_2,B]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = A_1[A_2,B]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + [A_1,B]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}A_2, \tag{7.46a}$$
$$[A,B_1B_2]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = B_1[A,B_2]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + [A,B_1]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B_2. \tag{7.46b}$$
   (a) Thus, evaluate, in two different ways,
   $$\begin{aligned}[A_1A_2,B_1B_2]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} &= A_1B_1[A_2,B_2]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + A_1[A_2,B_1]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B_2 \\ &\quad + B_1[A_1,B_2]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}A_2 + [A_1,B_1]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B_2A_2, \end{aligned} \tag{7.47a}$$
   $$\begin{aligned}[A_1A_2,B_1B_2]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} &= B_1A_1[A_2,B_2]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + B_1[A_1,B_2]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}A_2 \\ &\quad + A_1[A_2,B_1]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}B_2 + [A_1,B_1]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}A_2B_2. \end{aligned} \tag{7.47b}$$

(b) Subtracting these results, obtain

$$(A_1B_1 - B_1A_1)\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}(A_2B_2 - B_2A_2). \tag{7.48}$$

Thus, using the definition of the commutation relation,

$$\big[A, B\big] \equiv AB - BA, \tag{7.49}$$

obtain the relation

$$\big[A_1, B_1\big]\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}\big[A_2, B_2\big]. \tag{7.50}$$

(c) Since this condition holds for A_1 and B_1 independent of A_2 and B_2, conclude that

$$\big[A_1, B_1\big] = i\hbar\big[A_1, B_1\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \tag{7.51a}$$

$$\big[A_2, B_2\big] = i\hbar\big[A_2, B_2\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}}, \tag{7.51b}$$

where $i\hbar$ is necessarily a constant, independent of A_1, A_2, B_1, and B_2. This is the connection between the commutator braket in quantum mechanics and the Poisson braket in classical mechanics. If A's and B's are numbers, then, because their commutation relation is equal to zero, we necessairily have $\hbar = 0$. But, if the commutation relation of A's and B's is not zero, then finite values of $\hbar$ is allowed.

(d) Here the imaginary number $i = \sqrt{-1}$. Show that the constant $\hbar$ is a real number if we presume the Poisson braket to be real, and require the construction

$$C = \frac{1}{i}(AB - BA) \tag{7.52}$$

to be Hermitian. Experiment dictates that $\hbar = h/2\pi$, where

$$h \sim 6.63 \times 10^{-34}\,\text{J}\cdot\text{s} \tag{7.53}$$

is Planck's constant with dimensions of action.

7.3 Charge in a magnetic field

1. (**30 points.**) Hamiltonian for a charge particle of mass m and charge q in a magnetic field $\mathbf{B}$ is given by

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2, \tag{7.54}$$

where

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{7.55}$$

Let

$$\frac{\partial \mathbf{A}}{\partial t} = 0. \tag{7.56}$$

Further, the magnetic vector potential $\mathbf{A}(\mathbf{x}, t)$ is presumed to be independent of $\mathbf{p}$.

(a) Show that the Hamilton equations of motion leads to the equations, using ($\mathbf{v} = d\mathbf{x}/dt$)

$$m\mathbf{v} = \mathbf{p} - q\mathbf{A}, \tag{7.57a}$$

$$\frac{d\mathbf{p}}{dt} = q(\boldsymbol{\nabla}\mathbf{A}) \cdot \mathbf{v}. \tag{7.57b}$$

Show that the above equations in conjunction imply the familiar equation

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}. \tag{7.58}$$

(b) Evaluate the Poisson braket

$$\big[\mathbf{x}, \mathbf{x}\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{7.59}$$

(c) Evaluate the Poisson braket

$$\big[\mathbf{x}^i, \mathbf{v}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \frac{1}{m}\mathbf{1}^{ij}. \tag{7.60}$$

(d) Evaluate the Poisson braket

$$\big[\mathbf{x}^i, \mathbf{p}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \mathbf{1}^{ij}. \tag{7.61}$$

(e) Evaluate the Poisson braket

$$\big[m\mathbf{v}^i, m\mathbf{v}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = q(\boldsymbol{\nabla}^i\mathbf{A}^j - \boldsymbol{\nabla}^j\mathbf{A}^i). \tag{7.62}$$

Verify that

$$(\boldsymbol{\nabla}^i\mathbf{A}^j - \boldsymbol{\nabla}^j\mathbf{A}^i) = \varepsilon^{ijk}\mathbf{B}^k = -\mathbf{1}\times\mathbf{B}. \tag{7.63}$$

Poisson bracket in classical mechanics has direct correspondence to commutation relation in quantum mechanics through the factor $i\hbar$, which conforms with experiments and balances the dimensions. Then, we can write

$$\big[m\mathbf{v}^i, m\mathbf{v}^j\big] = i\hbar q\varepsilon^{ijk}\mathbf{B}^k \tag{7.64}$$

or

$$m\mathbf{v}\times m\mathbf{v} = i\hbar q\mathbf{B}, \tag{7.65}$$

using the fact that the commutator and the vector product satisfies the same Lie algebra as that of Poisson bracket.

(f) Evaluate the Poisson braket

$$\big[\mathbf{p}^i, \mathbf{v}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \frac{q}{m}\boldsymbol{\nabla}^i\mathbf{A}^j. \tag{7.66}$$

Using the antisymmetry property of the Poisson bracket conclude that

$$\big[\mathbf{p}^i, \mathbf{v}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[\mathbf{v}^i, \mathbf{p}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = \frac{q}{m}(\boldsymbol{\nabla}^i\mathbf{A}^j - \boldsymbol{\nabla}^j\mathbf{A}^i). \tag{7.67}$$

Thus, show that

$$\big[\mathbf{p}^i, \mathbf{v}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} + \big[\mathbf{v}^i, \mathbf{p}^j\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = -\frac{q}{m}\mathbf{1}\times\mathbf{B} \equiv \frac{q}{m}\varepsilon^{ijm}\mathbf{B}^m. \tag{7.68}$$

Deduce the corresponding expression in quantum mechanics to be

$$\mathbf{p}\times\mathbf{v} + \mathbf{v}\times\mathbf{p} = i\hbar\frac{q}{m}\mathbf{B}. \tag{7.69}$$

(g) Evaluate the Poisson braket

$$\big[\mathbf{p}, \mathbf{p}\big]^{\text{P.B.}}_{\mathbf{x},\mathbf{p}} = 0. \tag{7.70}$$

7.4 Infinitesimal canonical transformation

1. The generator for rotations satisfies the equations

$$\boldsymbol{\nabla}_{\mathbf{r}}G = -\delta\boldsymbol{\omega}\times\mathbf{p}, \tag{7.71a}$$
$$\boldsymbol{\nabla}_{\mathbf{p}}G = \delta\boldsymbol{\omega}\times\mathbf{r}. \tag{7.71b}$$

Show that

$$G = \delta\boldsymbol{\omega}\cdot\mathbf{L} \tag{7.72}$$

is a solution for the generator, where $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ is the angular momentum,

Chapter 8

Kepler problem

8.1 Ellipse

Refer Notes on Quantum Mechanics.

8.2 Conserved quantities

1. (**20 points.**) Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the postions of masses m_1 and m_2, respectively, with repect to an inertial frame. The gravitational interaction energy between the two masses is given by

$$\frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}. \tag{8.1}$$

Assume that the masses have no other internal or external interaction. The position of the center of mass $\mathbf{R}$ is defined by

$$(m_1 + m_2)\mathbf{R} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 \tag{8.2}$$

and the relative position between the masses is given by

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1. \tag{8.3}$$

What is the motion of the center of mass $\mathbf{R}$ with respect to the position $\mathbf{r}_1$.

(a) Stays fixed.

(b) Circular.

(c) Elliptic (or conic section).

(d) None of the above.

Hint: The positions represented by the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{R}$ are collinear. Further, $\mathbf{r}$ describes an ellipse.
Solution: Show that

$$\mathbf{R} - \mathbf{r}_1 = \frac{m_2}{m_1 + m_2}\mathbf{r}, \tag{8.4a}$$

$$\mathbf{R} - \mathbf{r}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{r}. \tag{8.4b}$$

Then, using $\mathbf{r}$ is elliptic, conclude that $\mathbf{R} - \mathbf{r}_i$ describes an ellipse whose length is scaled down by the factor $m_i/(m_1 + m_2)$.

2. (**20 points.**) (Refer Schwinger's QM, chapter 9) The Hamiltonian for a Kepler problem is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{\alpha}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{8.5}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the positions of the two constituent particles of masses m_1 and m_2.

(a) Introduce the coordinates representing the center of mass, relative position, total momentum, and relative momentum:

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r_1} - \mathbf{r_2}, \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad \mathbf{p} = \frac{m_2\mathbf{p}_1 - m_1\mathbf{p}_2}{m_1 + m_2}, \tag{8.6}$$

respectively, to rewrite the Hamiltonian as

$$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} - \frac{\alpha}{r}, \tag{8.7}$$

where

$$M = m_1 + m_2, \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \tag{8.8}$$

(b) Show that Hamilton's equations of motion are given by

$$\frac{d\mathbf{R}}{dt} = \frac{\mathbf{P}}{M}, \quad \frac{d\mathbf{P}}{dt} = 0, \quad \frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{\mu}, \quad \frac{d\mathbf{p}}{dt} = -\frac{\alpha\mathbf{r}}{r^3}. \tag{8.9}$$

(c) Verify that the Hamiltonian H, the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the Laplace-Runge-Lenz vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p} \times \mathbf{L}}{\mu\alpha}, \tag{8.10}$$

are the three constants of motion for the Kepler problem. That is, show that

$$\frac{dH}{dt} = 0, \quad \frac{d\mathbf{L}}{dt} = 0, \quad \frac{d\mathbf{A}}{dt} = 0. \tag{8.11}$$

3. (**20 points.**) The Hamiltonian for a Kepler problem (or a classical hydrogenic atom) is

$$H(\mathbf{r}, \mathbf{p}) = \frac{p^2}{2m} - \frac{\alpha}{r}, \tag{8.12}$$

where $r = |\mathbf{r}|$ and $p = |\mathbf{p}|$. The Hamilton's equations of motion for the Kepler are

$$\frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{m}, \qquad \frac{d\mathbf{p}}{dt} = -\alpha\frac{\mathbf{r}}{r^3}. \tag{8.13}$$

The Hamiltonian H, the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the Laplace-Runge-Lenz vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{1}{m\alpha}\mathbf{p} \times \mathbf{L}, \tag{8.14}$$

are the three constants of motion for a Kepler problem. Under the special circumstance when $r = |\mathbf{r}|$ is also a conserved quantity, that is,

$$\frac{dr}{dt} = 0, \tag{8.15}$$

we have the case of circular motion. Evaluate the Laplace-Runge-Lenz vector for this case of circular orbit.

4. (**50 points.**) The Hamiltonian for a Kepler problem is

$$H(\mathbf{r},\mathbf{p}) = \frac{p^2}{2\mu} - \frac{\alpha}{r}. \tag{8.16}$$

The Hamiltonian H, the angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, and the axial vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{\mathbf{p}\times\mathbf{L}}{\mu\alpha}, \tag{8.17}$$

are conserved quantities for a Kepler problem.

(a) Show that

$$\mathbf{W} = \frac{\mu\alpha}{L^2}\mathbf{A}\times\mathbf{L} \tag{8.18}$$

is also a conserved quantity. That is, show that $d\mathbf{W}/dt = 0$. Thus, together, the vectors $\mathbf{L}$, $\mathbf{A}$, and $\mathbf{W}$, form an orthogonal set that remain fixed in time. Show that the vector $\mathbf{W}$ can be expressed in the form

$$\mathbf{W} = \mathbf{p} + \frac{\mu\alpha}{L^2}\hat{\mathbf{r}}\times\mathbf{L}. \tag{8.19}$$

Further, show that

$$W = \mu\alpha\frac{A}{L}. \tag{8.20}$$

(b) Determine the components of the momentum $\mathbf{p}$ along these orthogonal vectors by evaluating $(\mathbf{p}\cdot\hat{\mathbf{L}})$, $(\mathbf{p}\cdot\hat{\mathbf{A}})$, and $(\mathbf{p}\cdot\hat{\mathbf{W}})$. Thus, construct the momentum $\mathbf{p}$ in the form

$$\mathbf{p} = (\mathbf{p}\cdot\hat{\mathbf{L}})\,\hat{\mathbf{L}} + (\mathbf{p}\cdot\hat{\mathbf{A}})\,\hat{\mathbf{A}} + (\mathbf{p}\cdot\hat{\mathbf{W}})\,\hat{\mathbf{W}}. \tag{8.21}$$

Hint: Show that

$$\mathbf{p}\cdot\mathbf{L} = 0, \qquad \mathbf{p}\cdot\mathbf{A} = \mathbf{p}\cdot\hat{\mathbf{r}}, \qquad \mathbf{p}\cdot\mathbf{W} = \frac{p^2}{2} + \mu H. \tag{8.22}$$

(c) It is well known that the position $\mathbf{r}$ traverses an ellipse about the origin. This is the content of Kepler's first law of motion. Show that the momentum $\mathbf{p}$ traverses a circle about a fixed point $\mathbf{p}_0$. That is, show that the momentum $\mathbf{p}$ satisfies the equation of a circle,

$$|\mathbf{p} - \mathbf{p}_0| = q. \tag{8.23}$$

Hint: Rewrite the expression for $(\mathbf{p}\cdot\hat{\mathbf{W}})$ in the form $\mathbf{p}\cdot\mathbf{p} - 2\mathbf{p}\cdot\mathbf{W} + \mathbf{W}\cdot\mathbf{W} = W^2 - 2\mu H$.

(d) Determine the vector $\mathbf{p}_0$ representing the center of this circle, and find the radius q of this circle. Verify that the center $\mathbf{p}_0$ is a conserved quantity.
Solution: $\mathbf{p}_0 = \mathbf{W}$ and $q = \mu\alpha/L$.

(e) Show that when the position $\mathbf{r}$ traverses a circle ($A = 0$) the center of the circle traversed by momentum $\mathbf{p}$ is the origin.

8.3 Kepler orbits

1. (**20 points.**) Starting from the Lagrangian for the Kepler problem,

$$L(\mathbf{r},\mathbf{v}) = \frac{1}{2}\mu v^2 + \frac{\alpha}{r}, \tag{8.24}$$

derive Kepler's first law of planetary motion, which states that the orbit of a planet is a conic section. In particular, derive

$$r(\phi) = \frac{r_0}{1 + e\cos(\phi - \phi_0)}, \tag{8.25}$$

which is the equation of a conic section in terms of the eccentricity e and a distance r_0. The distance r_0 is characterized by the fact that the effective potential

$$U_{\text{eff}}(r) = \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r} \tag{8.26}$$

is minimum at r_0. We used the definitions, $L_z = \mu r^2 \dot{\phi}$,

$$r_0 = \frac{L_z^2}{\mu\alpha}, \qquad U_{\text{eff}}(r_0) = -\frac{\alpha}{2r_0}, \qquad e = \sqrt{1 - \frac{E}{U_{\text{eff}}(r_0)}}. \tag{8.27}$$

Thus, the orbit of a planet is completely determined by the energy E and the angular momentum L_z, which are constants of motion.

2. (**20 points.**) In the Kepler problem the orbit of a planet is a conic section

$$r(\phi) = \frac{r_0}{1 + e\cos(\phi - \phi_0)} \tag{8.28}$$

expressed in terms of the eccentricity e and distance r_0. Determine the constant ϕ_0 to be 0 by requiring the initial condition

$$r(0) = \frac{r_0}{1+e}. \tag{8.29}$$

This leads to

$$r(\pi) = \frac{r_0}{1-e}. \tag{8.30}$$

The distance r_0 is characterized by the fact that the effective potential

$$U_{\text{eff}}(r) = \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r} \tag{8.31}$$

is minimum at r_0. We used the definitions

$$r_0 = \frac{L_z^2}{\mu\alpha}, \qquad U_{\text{eff}}(r_0) = -\frac{\alpha}{2r_0}, \qquad e = \sqrt{1 - \frac{E}{U_{\text{eff}}(r_0)}}. \tag{8.32}$$

Thus, the orbit of a planet is completely determined by the energy E and the angular momentum L_z, which are constants of motion. The statement of conservation of angular momentum can be expressed in the form

$$dt = \frac{\mu}{L_z} r^2 d\phi, \tag{8.33}$$

which is convenient for evaluating the time elapsed in the motion. For the case of elliptic orbit, $U_{\text{eff}}(r_0) < E < 0$, show that the time period is given by

$$T = \frac{\mu}{L_z}\int_0^{2\pi} d\phi \frac{r_0^2}{(1+e\cos\phi)^2} = \frac{\mu r_0^2}{L_z}\frac{2\pi}{(1-e^2)^{\frac{3}{2}}}. \tag{8.34}$$

Show that at point '2' in Figure 2

$$\phi = \frac{\pi}{2}, \quad \text{and} \quad r = r_0. \tag{8.35}$$

The time taken to go from '1' to '2' is given by (need not be proved here)

$$t_{1\to 2} = \frac{\mu}{L_z}\int_0^{\frac{\pi}{2}} d\phi \frac{r_0^2}{(1+e\cos\phi)^2} = \frac{T}{4}\left(\frac{4}{\pi}\tan^{-1}\sqrt{\frac{1-e}{1+e}} - \frac{2e}{\pi}\sqrt{1-e^2}\right). \tag{8.36}$$

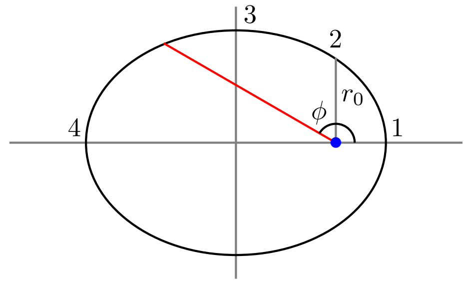

Figure 8.1: Elliptic orbit

Evaluate $t_{1\to 2}$ for $e = 0$ and $e = 1$. Show that at point '3' in Figure 2

$$\phi = \pi - \tan^{-1}\left(\frac{\sqrt{1-e^2}}{e}\right), \quad \text{and} \quad r = a. \tag{8.37}$$

The time taken to go from '1' to '3' is given by (need not be proved here)

$$t_{1\to 3} = \frac{\mu}{L_z}\int_0^{\pi-\tan^{-1}\left(\frac{\sqrt{1-e^2}}{e}\right)} d\phi \frac{r_0^2}{(1+e\cos\phi)^2} = \frac{T}{4}\left(1 - \frac{2e}{\pi}\right). \tag{8.38}$$

Similarly, the time taken to go from '3' to '4' is given by (need not be proved here)

$$t_{3\to 4} = \frac{\mu}{L_z}\int_{\pi-\tan^{-1}\left(\frac{\sqrt{1-e^2}}{e}\right)}^{\pi} d\phi \frac{r_0^2}{(1+e\cos\phi)^2} = \frac{T}{4}\left(1 + \frac{2e}{\pi}\right). \tag{8.39}$$

Evaluate the time elapsed in the above cases for $e \to 0$ and $e \to 1$. The eccentricity e of Earth's orbit is 0.0167 and timeperiod T is 365 days. Thus, calculate

$$t_{1\to 3} - t_{1\to 2} \tag{8.40}$$

for Earth in units of days.
Solution: ~ 1 day for Earth.

3. (**20 points.**) Refer to the essay by J. M. Luttinger titled 'On "negative" mass in the theory of gravitation' in 1951.

   (a) Reproduce all the equations in the essay.

   (b) Critically assess the logic of the arguments in the essay.

8.4 Precession of the perihelion

1. (**20 points.**) The effective potential energy for the Kepler problem is

$$U_{\text{eff}}(r) = \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r}, \tag{8.41}$$

where the first term is the energy associated with the centrifugal force and the second term is the gravitational potential energy. Show that the equilibrium point for the above potential energy function is

$$r_0 = \frac{L_z^2}{\mu\alpha} \tag{8.42}$$

and the corresponding minimum energy is

$$U_{\text{eff}}(r_0) = -\frac{\alpha}{2r_0}. \tag{8.43}$$

For total energy $E < 0$ show that the potential energy function have two turning points,

$$r_{\text{min}} = \frac{r_0}{1+e} \tag{8.44}$$

and

$$r_{\text{max}} = \frac{r_0}{1-e}, \tag{8.45}$$

where the eccentricity e is given by

$$e = \sqrt{1 - \frac{E}{U_{\text{eff}}(r_0)}}. \tag{8.46}$$

Next, consider a perturbation to the effective potential energy,

$$U'_{\text{eff}}(r) = \frac{L_z^2}{2\mu r^2} - \frac{\alpha}{r} + \frac{\beta_3}{r^3}, \tag{8.47}$$

such that

$$\kappa = \frac{\beta_3/r_0^3}{\alpha/r_0} = \frac{\beta_3}{\alpha r_0^2} \ll 1. \tag{8.48}$$

To the leading order in κ, show that the shift in the equilibrium point is

$$r'_0 = r_0(1+3\kappa) \tag{8.49}$$

and the leading order shift in the minimum energy is

$$U'_{\text{eff}}(r'_0) = U_{\text{eff}}(r_0)\Big[1 - 2\kappa\Big]. \tag{8.50}$$

Show that the leading order shifts in the turning points are

$$r'_{\text{min}} = r_{\text{min}}\left[1 + \kappa\frac{(1+e)^2}{e}\right] \tag{8.51}$$

and

$$r'_{\text{max}} = r_{\text{max}}\left[1 - \kappa\frac{(1-e)^2}{e}\right]. \tag{8.52}$$

After the perturbation the trajectory is no more an ellipse. Nevertheless, for small perturbation we can define the leading order shift in the eccentricity using

$$e' = \frac{r'_{\text{max}} - r'_{\text{min}}}{r'_{\text{max}} + r'_{\text{min}}}. \tag{8.53}$$

Evaluate

$$e' = e\left[1 - \kappa\frac{(1-e^2)^2}{e^2}\right]. \tag{8.54}$$

Illustrate the above shifts in the plot for effective potential energy.

2. (**20 points.**) (Resource: Lecture from [2024S].) Kepler problem is described by the potential energy

$$U(r) = -\frac{\alpha}{r}, \tag{8.55}$$

and the corresponding Lagrangian

$$L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}\mu v^2 + \frac{\alpha}{r}. \tag{8.56}$$

The angular momentum

$$L_z = \mu r^2 \dot{\phi} \tag{8.57}$$

and the energy

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{L_z^2}{\mu r^2} - \frac{\alpha}{r} \tag{8.58}$$

and constants of motion in the Kepler problem. For the case when the energy E is negative,

$$-\frac{\alpha}{2r_0} < E < 0, \qquad r_0 = \frac{L_z^2}{\mu\alpha}, \tag{8.59}$$

where L_z is the angular momentum, the motion is described by an ellipse,

$$r(\phi) = \frac{r_0}{1 + e\cos(\phi - \phi_0)}, \qquad e = \sqrt{1 + \frac{E}{(\alpha/2r_0)}}. \tag{8.60}$$

Perihelion is the point in the orbit of a planet when it is closest to the Sun. This corresponds to $\phi = \phi_0$. The precession of the perihelion is suitably defined in terms of the angular displacement $\Delta\phi$ of the perihelion during one revolution,

$$\Delta\phi = 2\left[\int_{r_{\min}}^{r_{\max}} d\phi\right] - 2\pi, \tag{8.61}$$

where one revolution is defined as twice the transition between points when the planet is closest and farthest from Sun in terms of

$$r_{\min} = \frac{r_0}{1+e} \tag{8.62}$$

the perihelion, when the planet is closest to Sun, and

$$r_{\max} = \frac{r_0}{1-e} \tag{8.63}$$

is the aphelion, corresponding to $\phi = \phi_0 + \pi$, when the planet is farthest from Sun.

(a) For the Kepler problem, starting from the expression for energy and angular momentum, derive the relation

$$d\phi = \frac{r_0\, dr}{r^2}\frac{1}{\sqrt{e^2 - \left(1 - \frac{r_0}{r}\right)^2}}. \tag{8.64}$$

The precession of perihelion is zero for the Kepler problem. Show this by evaluating

$$\Delta\phi = 2\left[\int_{\frac{r_0}{1+e}}^{\frac{r_0}{1-e}} \frac{r_0\, dr}{r^2}\frac{1}{\sqrt{e^2 - \left(1 - \frac{r_0}{r}\right)^2}}\right] - 2\pi. \tag{8.65}$$

This is easily achieved by substituting

$$1 - \frac{r_0}{r} = -e\cos\theta, \tag{8.66}$$

with the associated differntial statement

$$\frac{r_0 dr}{r^2} = e\sin\theta d\theta, \tag{8.67}$$

and the related changes in the limits of integration

$$r = \frac{r_0}{1+e} \quad \rightarrow \quad \theta = 0, \tag{8.68}$$

$$r = \frac{r_0}{1-e} \quad \rightarrow \quad \theta = \pi, \tag{8.69}$$

to obtain

$$\Delta\phi = 2\left[\int_0^\pi d\theta\right] - 2\pi = 0. \tag{8.70}$$

(b) When a small correction

$$\delta U(r) = -\frac{\beta}{r^3} = \kappa\, U_0 \left(\frac{r_0}{r}\right)^3, \tag{8.71}$$

expressed in terms of dimensionless parameter κ using the relation $\beta = -\kappa U_0 r_0^3$, is added we have the perturbed potential energy

$$U(r) = -\frac{\alpha}{r} - \frac{\beta}{r^3} = -\frac{\alpha}{2r_0}\left[\frac{r_0}{r} + \kappa\left(\frac{r_0}{r}\right)^3\right]. \tag{8.72}$$

Show that the precession of the perihelion due to this perturbation is

$$\Delta\phi = -3\pi\kappa = -\frac{6\pi\beta}{\alpha r_0^2}. \tag{8.73}$$

Chapter 9

Special Relativity

9.1 Relativity principle

Problems

1. (**20 points.**) The relativity principle states that the laws of physics are invariant (or covariant) when observed using different coordinate systems. In special relativity we restrict these coordinate systems to be uniformly moving with respect to each other. Let $z = z' = 0$ at $t = 0$.

   (a) Linear: Spatial homogeneity, spatial isotropy, and temporal homogeneity, require the transformation to be linear. (We will skip this derivation.) Then, for simplicity, restricting to coordinate systems moving with respect to each other in a single direction, we can write

   $$z' = A(v)\, z + B(v)\, t, \tag{9.1a}$$
   $$t' = E(v)\, z + F(v)\, t. \tag{9.1b}$$

   We will refer to the respective frames as primed and unprimed.

   (b) Identity: An object P at rest in the primed frame, described by $z' = 0$, will be described in the unprimed frame as $z = vt$.

   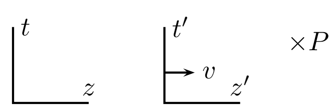

   Figure 9.1: Identity.

   Using these in Eq. (9.1a), we have

   $$0 = A(v)\, vt + B(v)\, t. \tag{9.2}$$

   This implies $B(v) = -vA(v)$. Thus, show that

   $$z' = A(v)\,(z - vt), \tag{9.3a}$$
   $$t' = E(v)\, z + F(v)\, t. \tag{9.3b}$$

   (c) Reversal: The descriptions of a process in the unprimed frame moving to the right with velocity v with respect to the primed should be identical to those made in the unprimed (with their axis flipped)

Figure 9.2: Reversal.

moving with velocity $-v$ with respect to the primed (with their axis flipped). This is equivalent to the requirement of isotropy in an one dimensional space.
That is, the transformation must be invariant under

$$z \to -z, \qquad z' \to -z', \qquad v \to -v. \tag{9.4}$$

This implies

$$-z' = A(-v)\,(-z + vt), \tag{9.5a}$$
$$t' = -E(-v)\,z + F(-v)\,t. \tag{9.5b}$$

Show that Eqs. (9.3a) and (9.5a) in conjunction imply

$$A(-v) = A(v). \tag{9.6}$$

Further, show that Eqs. (9.3b) and (9.5b) in conjunction implies

$$E(-v) = -E(v), \tag{9.7a}$$
$$F(-v) = F(v). \tag{9.7b}$$

(d) Reciprocity: The description of a process in the unprimed frame moving to the right with velocity v is identical to the description in the primed frame moving to the left.

Figure 9.3: Reciprocity.

That is, the transformation must be invariant under

$$(z,t) \to (z',t') \qquad (z',t') \to (z,t) \qquad v \to -v. \tag{9.8}$$

Show that this implies

$$z = A(-v)\,(z' + vt'), \tag{9.9a}$$
$$t = E(-v)\,z' + F(-v)\,t'. \tag{9.9b}$$

Show that Eqs. (9.3) and Eqs. (9.9) imply

$$E(v) = \frac{1}{v}\left[\frac{1}{A(v)} - A(v)\right], \tag{9.10a}$$
$$F(v) = A(v). \tag{9.10b}$$

(e) Together, for arbitrary $A(v)$, show that the relativity principle allows the following transformations,

$$z' = A(v)\,(z - vt), \tag{9.11a}$$

$$t' = A(v)\left[\frac{1}{v}\left(\frac{1}{A(v)^2} - 1\right) z + t\right]. \tag{9.11b}$$

i. In Galilean relativity we require $t' = t$. Show that this is obtained with

$$A(v) = 1 \tag{9.12}$$

in Eqs. (9.11). This leads to the Galilean transformation

$$z' = z - vt, \tag{9.13a}$$

$$t' = t. \tag{9.13b}$$

ii. In Einstein's special relativity the requirement is for a special speed c that is described identically by both the primed and unprimed frames. That is,

$$z = ct, \tag{9.14a}$$

$$z' = ct'. \tag{9.14b}$$

Show that Eqs. (9.14) when substituted in in Eqs. (9.11) leads to

$$A(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{9.15}$$

This corresponds to the Lorentz transformation

$$z' = A(v)(z - vt), \tag{9.16a}$$

$$t' = A(v)\left(-\frac{v}{c^2} z + t\right). \tag{9.16b}$$

iii. This suggests that it should be possible to contrive additional solutions for $A(v)$ that respects the relativity principle, but with new physical requirements for the respective choice of $A(v)$. Construct one such transformation. In particular, investigate modifications of Eqs. (9.14) that donot change the current experimental observations. The response to this part of the question will not be used for assessment.

9.2 Lorentz transformation

Problems

1. (**20 points.**) The Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \qquad \beta = \frac{v}{c}. \tag{9.17}$$

(a) Evaluate γ for $v = 30\,\text{m/s}$ ($\sim 70\,\text{miles/hour}$).

(b) Evaluate γ for $v = 3c/5$.

2. (**20 points.**) Lorentz transformation describing a boost in the x-direction is obtained using the matrix

$$L = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9.18}$$

(a) Show that the determinant of the matrix L is 1.

(b) Determine L^{-1}.

3. (**20 points.**) Lorentz transformation (in one dimension) is given by

$$\Delta z' = \gamma(\Delta z - v\Delta t), \tag{9.19a}$$

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c}\frac{\Delta z}{c}\right), \tag{9.19b}$$

where $\gamma = \sqrt{1 - v^2/c^2}$. Show that for

$$v \ll c \qquad \text{and} \qquad \frac{\Delta z}{\Delta t} \ll c \tag{9.20}$$

one obtains the Galilean transformation

$$\Delta z' = \Delta z - v\Delta t, \tag{9.21a}$$

$$\Delta t' = \Delta t. \tag{9.21b}$$

Note: For the case when Δz and Δt represent the change in position and time of a particle we could have v and $\Delta z/\Delta t$ to be identical.

4. (**20 points.**) How does the wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f(z - ct) = 0 \tag{9.22}$$

transform under the Lorentz transformtion

$$z' = \gamma z + \beta\gamma ct, \tag{9.23a}$$

$$ct' = \beta\gamma z + \gamma ct. \tag{9.23b}$$

Solution:

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f(a(z - ct)) = 0, \tag{9.24}$$

where $a = \sqrt{(1-\beta)/(1+\beta)}$.

5. (**20 points.**) Verify the following:

$$\text{Tr}A = A_i{}^i. \tag{9.25a}$$

$$\det A = \varepsilon_{i_1 i_2 \ldots i_n} A^{i_1}{}_1 A^{i_2}{}_2 \ldots A^{i_n}{}_n \tag{9.25b}$$

$$= \frac{1}{n!}\varepsilon_{i_1 i_2 \ldots i_n} \varepsilon^{i'_1 i'_2 \ldots i'_n} A^{i_1}{}_{i'_1} A^{i_2}{}_{i'_2} \ldots A^{i_n}{}_{i'_n}, \tag{9.25c}$$

where n is the dimension of the matrix A.

6. (**20 points.**) Prove that any orthogonal matrix R satisfying

$$RR^T = 1 \tag{9.26}$$

in N-dimensions has $N(N-1)/2$ independent variables.

7. (**20 points.**) Lorentz transformation describing a boost in the x-direction, y-direction, and z-direction, are

$$L_1 = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} \gamma_2 & 0 & -\beta_2\gamma_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_2\gamma_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} \gamma_3 & 0 & 0 & -\beta_3\gamma_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_3\gamma_3 & 0 & 0 & \gamma_3 \end{pmatrix}, \tag{9.27}$$

respectively. Transformation describing a rotation about the x-axis, y-axis, and z-axis, are

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\omega_1 & \sin\omega_1 \\ 0 & 0 & -\sin\omega_1 & \cos\omega_1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_2 & 0 & -\sin\omega_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\omega_2 & 0 & \cos\omega_2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_3 & \sin\omega_3 & 0 \\ 0 & -\sin\omega_3 & \cos\omega_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{9.28}$$

respectively. For infinitesimal transformations, $\beta_i = \delta\beta_i$ and $\omega_i = \delta\omega_i$ use the approximations

$$\gamma_i \sim 1, \qquad \cos\omega_i \sim 1, \qquad \sin\omega_i \sim \delta\omega_i, \tag{9.29}$$

to identify the generator for boosts $\mathbf{N}$, and the generator for rotations the angular momentum $\mathbf{J}$,

$$\mathbf{L} = \mathbf{1} + \delta\boldsymbol{\beta}\cdot\mathbf{N} \qquad \text{and} \qquad \mathbf{R} = \mathbf{1} + \delta\boldsymbol{\omega}\cdot\mathbf{J}, \tag{9.30}$$

respectively. Then derive

$$\Big[N_1, N_2\Big] = N_1N_2 - N_2N_1 = J_3. \tag{9.31}$$

This states that boosts in perpendicular direction leads to rotation. (To gain insight of the statement, calculate $[J_1, J_2]$ and interpret the result.)

(a) Is velocity addition commutative?

(b) Is velocity addition associative?

(c) Read a resource article (Wikipedia) on Wigner rotation.

8. (**20 points.**) (Based on Hughston and Tod's book.) Prove the following.

(a) If p^μ is a time-like vector and $p^\mu s_\mu = 0$ then s^μ is necessarily space-like.

(b) If p^μ and q^μ are both time-like vectors and $p^\mu q_\mu < 0$ then either both are future-pointing or both are past-pointing.

(c) If p^μ and q^μ are both light-like vectors and $p^\mu q_\mu = 0$ then p^μ and q^μ are proportional.

(d) If p^μ is a light-like vector and $p^\mu s_\mu = 0$, then s^μ is space-like or p^μ and s^μ are proportional.

(e) If u^α, v^α, and w^α, are time-like vectors with $u^\alpha v_\alpha < 0$ and $v^\alpha w_\alpha < 0$, then $w^\alpha u_\alpha < 0$.

9. (**20 points.**) Non-relativistic limits are obtained for $\beta \ll 1$ in relativistic formulae.

(a) Does Lorentz transformation recover Galilean transformation for $\beta \ll 1$?

(b) Does Lorentz transformation recover Galilean transformation for $\beta \ll 1$ and $c \to \infty$?

9.3 Geometry of Lorentz transformation

1. (**20 points.**) A four-vector in the context of Lorentz tranformation can be described using the notation

$$a^\alpha = (a^0, a^1, a^2, a^3). \tag{9.32}$$

Let

$$b^\alpha = (b^0, b^1, b^2, b^3) \tag{9.33}$$

be another four-vector. The scalar product between two Lorentz vectors is given by

$$a^\alpha b_\alpha = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3. \tag{9.34}$$

The square of the 'length' of the four-vector a^α is given by

$$a^\alpha a_\alpha, \tag{9.35}$$

which is not necessarily positive. The length of a four-vector is invariant, that is, it is independent of the Lorentz frame. If two Lorentz four-vectors are orthogonal they satisfy

$$a^\alpha b_\alpha = 0. \tag{9.36}$$

Orthogonality is an invariant concept.

(a) Determine the length of

$$p^\alpha = (5, 0, 0, 3), \tag{9.37}$$

where the numbers are in arbitrary units. Is it time-like, light-like, or space-like?

(b) Find a four-vector of the form

$$q^\alpha = (q^0, 0, 0, q^3) \tag{9.38}$$

that is perpendicular to p^α.

2. (**20 points.**) A hypothetical particle is observed by an inertial observer to be moving with non-uniform superluminal speed ($v > c$) at every instant of time from remote past to remote future. Draw a plausible world line of such a particle.

9.4 Poincaré (parallel) velocity addition formula

1. (**20 points.**) The Poincaré formula for the addition of (parallel) velocities is

$$v = \frac{v_a + v_b}{1 + \frac{v_a v_b}{c^2}}, \tag{9.39}$$

where v_a and v_b are velocities and c is speed of light in vacuum. Jerzy Kocik, from the department of Mathematics in SIUC, has invented a geometric diagram that allows one to visualize the Poincaré formula. (Refer [2].) An interactive applet for exploring velocity addition is available at Kocik's web page [1]. (For the following assume that the Poincaré formula holds for all speeds, subluminal ($v_i < c$), superluminal ($v_i > c$), and speed of light.)

(a) Analyse what is obtained if you add two subluminal speeds?

(b) Analyse what is obtained if you add a subluminal speed to speed of light?

(c) Analyse what is obtained if you add a subluminal speed to a superluminal speed?

(d) Analyse what is obtained if you add speed of light to another speed of light?

(e) Analyse what is obtained if you add a superluminal speed to speed of light?

(f) Analyse what is obtained if you add two superluminal speeds?

2. (**20 points.**) The Poincaré formula for the addition of (parallel) velocities is, $c = 1$,

$$v = \frac{v_a + v_b}{1 + v_a v_b}, \tag{9.40}$$

where v_a and v_b are velocities and c is speed of light in vacuum. Assuming that the Poincaré formula holds for all speeds, subluminal ($-1 < v_i < 1$), superluminal ($|v_i| > 1$), and speed of light, analyse what is obtained if you add a subluminal speed to a superluminal speed? That is, is the 'sum' subluminal or superluminal. Is the answer unique?

3. (**20 points.**) The Poincaré formula for the addition of (parallel) velocities is

$$v = \frac{v_a + v_b}{1 + \dfrac{v_a v_b}{c^2}}, \tag{9.41}$$

where v_a and v_b are velocities and c is speed of light in vacuum. (For the following assume that the Poincaré formula holds for all speeds, subluminal ($v_i < c$), superluminal ($v_i > c$), and speed of light.) Analyse what is obtained if you add a subluminal speed to a superluminal speed? That is, is the resultant speed subluminal or superluminal.
Hint: Analyse the case

$$\frac{v_a}{c} = -\frac{c}{v_b} \pm \delta, \tag{9.42}$$

for infinitely small $\delta > 0$.

4. (**20 points.**) The Poincaré formula for the addition of (parallel) velocities is, $c = 1$,

$$v = \frac{v_a + v_b}{1 + v_a v_b}, \tag{9.43}$$

where v_a and v_b are velocities and c is speed of light in vacuum. Assuming that the Poincaré formula holds for all speeds, subluminal ($-1 < v_i < 1$), superluminal ($|v_i| > 1$), and speed of light, analyse what is obtained if you add a speed to an infinitely large superluminal speed, that is, $v_b \to \infty$. Hint: Inversion.

5. (**30 points.**) Let

$$\tanh\theta = \beta, \tag{9.44}$$

where $\beta = v/c$. Addition of (parallel) velocities in terms of the parameter θ obeys the arithmatic addition

$$\theta = \theta_a + \theta_b. \tag{9.45}$$

   (a) Invert the expression in Eq. (9.44) to find the explicit form of θ in terms of β as a logarithm.
   (b) Show that Eq. (9.45) leads to the relation

$$\left(\frac{1+\beta}{1-\beta}\right) = \left(\frac{1+\beta_a}{1-\beta_a}\right)\left(\frac{1+\beta_b}{1-\beta_b}\right). \tag{9.46}$$

   (c) Using Eq. (9.46) derive the Poincaré formula for the addition of (parallel) velocities.

9.5 Kinematics

1. (**100 points.**) Relativisitic kinematics is constructed in terms of the proper time element ds, which remains unchanged under a Lorentz transformation,

$$-ds^2 = -c^2 dt^2 + d\mathbf{x} \cdot d\mathbf{x}. \tag{9.47}$$

Here $\mathbf{x}$ and t are the position and time of a particle. They are components of a vector under Lorentz transformation and together constitute the position four-vector

$$x^\alpha = (ct, \mathbf{x}). \tag{9.48}$$

(a) Velocity: The four-vector associated with velocity is constructed as

$$u^\alpha = c\frac{dx^\alpha}{ds}. \tag{9.49}$$

i. Using Eq. (9.47) deduce

$$\gamma ds = cdt, \quad \text{where} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \quad \mathbf{v} = \frac{d\mathbf{x}}{dt}. \tag{9.50}$$

Then, show that

$$u^\alpha = (c\gamma, \mathbf{v}\gamma). \tag{9.51}$$

Here $\mathbf{v}$ is the velocity that we use in Newtonian physics.

ii. Further, show that

$$u^\alpha u_\alpha = -c^2. \tag{9.52}$$

Thus, conclude that the velocity four-vector is a time-like vector. What is the physical implication of this statement for a particle?

iii. Write down the form of the velocity four-vector in the rest frame of the particle?

(b) Momentum: Define momentum four-vector in terms of the mass m of the particle as

$$p^\alpha = mu^\alpha = (mc\gamma, m\mathbf{v}\gamma). \tag{9.53}$$

Connection with the physical quantities associated to a moving particle, the energy and momentum of the particle, is made by identifying (or defining)

$$p^\alpha = \left(\frac{E}{c}, \mathbf{p}\right), \tag{9.54}$$

which corresponds to the definitions

$$E = mc^2\gamma, \tag{9.55a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{9.55b}$$

for energy and momentum, respectively. Discuss the non-relativistic limits of these quantities. In particular, using the approximation

$$\gamma = 1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots, \tag{9.56}$$

show that

$$E - mc^2 = \frac{1}{2}mv^2 + \ldots, \tag{9.57a}$$

$$\mathbf{p} = m\mathbf{v} + \ldots. \tag{9.57b}$$

Evaluate

$$p^\alpha p_\alpha = -m^2c^2. \tag{9.58}$$

Thus, derive the energy-momentum relation

$$E^2 - p^2c^2 = m^2c^4. \tag{9.59}$$

(c) Acceleration: The four-vector associated with acceleration is constructed as

$$a^\alpha = c\frac{du^\alpha}{ds}. \tag{9.60}$$

i. Show that
$$a^\alpha = \gamma \left(c\frac{d\gamma}{dt}, \mathbf{v}\frac{d\gamma}{dt} + \gamma \mathbf{a} \right), \tag{9.61}$$
where
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{9.62}$$
is the acceleration that we use in Newtonian physics.

ii. Starting from Eq. (9.52) and taking derivative with respect to proper time show that
$$u^\alpha a_\alpha = 0. \tag{9.63}$$
Thus, conclude that four-acceleration is space-like.

iii. Further, using the explicit form of $u^\alpha a_\alpha$ in Eq. (9.63) derive the identity
$$\frac{d\gamma}{dt} = \left(\frac{\mathbf{v} \cdot \mathbf{a}}{c^2} \right) \gamma^3. \tag{9.64}$$

iv. Show that
$$a^\alpha = \left(\frac{\mathbf{v} \cdot \mathbf{a}}{c} \gamma^4, \mathbf{a}\gamma^2 + \frac{\mathbf{v}}{c} \frac{\mathbf{v} \cdot \mathbf{a}}{c} \gamma^4 \right) \tag{9.65}$$

v. Write down the form of the acceleration four-vector in the rest frame ($\mathbf{v} = 0$) of the particle as $(0, \mathbf{a}_0)$, where
$$\mathbf{a}_0 = \mathbf{a}\big|_{\text{rest frame}} \tag{9.66}$$
is defined as the proper acceleration. Note that the proper acceleration is a Lorentz invariant quantity, that is, independent of which observer makes the measurement.

vi. Evaluate the following identities involving the proper acceleration
$$a^\alpha a_\alpha = \mathbf{a}_0 \cdot \mathbf{a}_0 = \left[\mathbf{a} \cdot \mathbf{a} + \left(\frac{\mathbf{v} \cdot \mathbf{a}}{c} \right)^2 \gamma^2 \right] \gamma^4 = \left[\mathbf{a} \cdot \mathbf{a} - \left(\frac{\mathbf{v} \times \mathbf{a}}{c} \right)^2 \right] \gamma^6. \tag{9.67}$$

vii. In a particular frame, if $\mathbf{v} \,||\, \mathbf{a}$ (corresponding to linear motion), deduce
$$|\mathbf{a}_0| = |\mathbf{a}| \gamma^3. \tag{9.68}$$
And, in a particular frame, if $\mathbf{v} \perp \mathbf{a}$ (corresponding to circular motion), deduce
$$|\mathbf{a}_0| = |\mathbf{a}| \gamma^2. \tag{9.69}$$

(d) Force: The force four-vector is defined as
$$f^\alpha = c\frac{dp^\alpha}{ds} = \left(\frac{\gamma}{c} \frac{dE}{dt}, \mathbf{F}\gamma \right), \tag{9.70}$$
where the force $\mathbf{F}$, identified (or defined) as
$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \tag{9.71}$$
is the force in Newtonian physics. Starting from Eq. (9.58) derive the relation
$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v} \tag{9.72}$$
which is the power output or the rate of work done by the force $\mathbf{F}$ on the particle.

(e) Equations of motion: The relativistic generalization of Newton's laws are

$$f^\alpha = ma^\alpha. \tag{9.73}$$

Show that these involve the relations, using the definitions of energy and momentum in Eqs. (9.55),

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}\gamma + m\mathbf{v}\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\gamma^3, \tag{9.74a}$$

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v} = m\mathbf{v}\cdot\mathbf{a}\gamma^3. \tag{9.74b}$$

Discuss the non-relativistic limits of the equations of motion.

2. (**20 points.**) Lorentz transformation relates the energy E and momentum $\mathbf{p}$ of a particle when measured in different frames. For example, for the special case when the relative velocity and the velocity of the particle are parallel we have

$$\begin{pmatrix} E'/c \\ p' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E/c \\ p \end{pmatrix}. \tag{9.75}$$

Photons are massless spin 1 particles whose energy and momentum are $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$, such that $\omega = kc$. Thus, derive the relativistic Doppler effect formula

$$\omega' = \omega\sqrt{\frac{1+\beta}{1-\beta}}. \tag{9.76}$$

Contrast the above formula with the Doppler effect formula for sound.

3. (**20 points.**) Neutral π meson decays into two photons. That is,

$$\pi^0 \to \gamma_1 + \gamma_2. \tag{9.77}$$

Energy-momentum conservation for the decay in the laboratory frame, in which the meson is not necessarily at rest, is given by

$$p_\pi^\alpha = p_1^\alpha + p_2^\alpha. \tag{9.78}$$

Or, more specifically,

$$\left(\frac{E_\pi}{c}, \mathbf{p}\right) = \left(\frac{E_1}{c}, \mathbf{p}_1\right) + \left(\frac{E_2}{c}, \mathbf{p}_2\right), \tag{9.79}$$

where E_π and $\mathbf{p}$ are the energy and momentum of neutral π meson, and E_i's and $\mathbf{p}_i$'s are the energies and momentums of the photons. Thus, derive the relation

$$m_\pi^2 c^4 = 2E_1E_2(1-\cos\theta), \tag{9.80}$$

where m_π is the mass of neutral π meson, and θ is the angle between the directions of $\mathbf{p}_1$ and $\mathbf{p}_2$.

4. (**20 points.**) Using Maxwell's equations we can show that a monochromatic electromagnetic wave has the electromagnetic energy density U and electromagnetic momentum density $\mathbf{G}$ given by

$$U = \frac{1}{2}\varepsilon_0^2E^2 + \frac{1}{2}\mu_0^2H^2 = \varepsilon_0^2E^2 = \mu_0^2H^2, \tag{9.81}$$

$$\mathbf{G} = \frac{\mathbf{E}\times\mathbf{H}}{c^2} = \hat{\mathbf{k}}\frac{U}{c}. \tag{9.82}$$

Observe that are densities. The energy and momentum densities do not transform like a four-vector, instead they are part of a four-tensor,

$$t^{\alpha\beta} = \begin{pmatrix} cU, & \mathbf{S} \\ c^2\mathbf{G}, & c\mathbf{T} \end{pmatrix}. \tag{9.83}$$

Note: Complete this!

5. (**20 points.**) Length contracts and time dilates. That is,

$$L = \frac{L_0}{\gamma}, \qquad T = T_0 \gamma, \tag{9.84}$$

where L_0 and T_0 are proper length and proper time. Similarly, show that (for $\mathbf{v} \,||\, \mathbf{a}$)

$$|\mathbf{a}| = \frac{|\mathbf{a}_0|}{\gamma^3}, \tag{9.85}$$

where $|\mathbf{a}_0|$ is the proper acceleration measured in the instantaneaous rest frame of the particle. Further, for $\mathbf{v} \perp \mathbf{a}$ show that

$$|\mathbf{a}| = \frac{|\mathbf{a}_0|}{\gamma^2}. \tag{9.86}$$

6. (**20 points.**) Time dilates. That is,

$$T = T_0 \gamma, \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{9.87}$$

where T_0 is the proper time measured in the instantaneous rest frame of the clock measuring T_0 and T is the time measured by a clock moving with velocity v relative to the clock measuring proper time. Similarly, show that (for $\mathbf{v} \,||\, \mathbf{a}$)

$$|\mathbf{a}| = \frac{|\mathbf{a}_0|}{\gamma^3}, \tag{9.88}$$

where $|\mathbf{a}_0|$ is the proper acceleration measured in the instantaneous rest frame of the particle. Derive the equation for the trajectory of a particle moving in a straight line (along the z axis) with constant proper acceleration, after starting from rest from the point $z = c^2/|\mathbf{a}_0|$ at time $t = 0$.

9.6 Dynamics

9.6.1 Charge particle in a uniform magnetic field: Circular motion

1. (**20 points.**) A relativisitic particle in a uniform magnetic field is described by the equations

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v}, \tag{9.89a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{9.89b}$$

where

$$E = mc^2 \gamma, \tag{9.90a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{9.90b}$$

and

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \tag{9.91}$$

Show that

$$\frac{d\gamma}{dt} = 0. \tag{9.92}$$

Then, derive

$$\frac{d\mathbf{v}}{dt} = \mathbf{v} \times \boldsymbol{\omega}_c, \tag{9.93}$$

where

$$\boldsymbol{\omega}_c = \frac{q\mathbf{B}}{m\gamma}. \tag{9.94}$$

Compare this relativistic motion to the associated non-relativistic motion.

2. (**20 points.**) If the motion of a non-relativistic particle is such that it does not change the kinetic energy of the particle, we have

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = 0. \tag{9.95}$$

Show that this imples

$$\mathbf{v} \cdot \mathbf{a} = 0. \tag{9.96}$$

This is achieved when the acceleration $a = 0$ or in the case of uniform circular motion. Starting from Eq. (9.96) show that the relativistic generalization of kinetic energy $E = mc^2\gamma$ is also conserved, that is,

$$\frac{d}{dt}(mc^2\gamma) = 0. \tag{9.97}$$

Observe that

$$\boldsymbol{\beta} \cdot \mathbf{a} = \frac{d}{dt}\left(\frac{\beta^2}{2}\right) = -\frac{1}{2}\frac{d}{dt}\frac{1}{\gamma^2} = \frac{1}{\gamma^3}\frac{d\gamma}{dt}. \tag{9.98}$$

9.6.2 Charge particle in a uniform electric field: Hyperbolic motion

1. (**20 points.**) A relativisitic particle in a uniform electric field is described by the equations

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v}, \tag{9.99a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{9.99b}$$

where

$$E = mc^2\gamma, \tag{9.100a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{9.100b}$$

and

$$\mathbf{F} = q\mathbf{E}. \tag{9.101}$$

Let us consider the configuration with the electric field in the $\hat{\mathbf{y}}$ direction,

$$\mathbf{E} = E\,\hat{\mathbf{y}}, \tag{9.102}$$

and initial conditions

$$\mathbf{v}(0) = 0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{9.103a}$$

$$\mathbf{x}(0) = 0\,\hat{\mathbf{x}} + y_0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}. \tag{9.103b}$$

(a) In terms of the definition

$$\boldsymbol{\omega}_0 = \frac{1}{c}\frac{q\mathbf{E}}{m}, \tag{9.104}$$

show that the equations of motion are given by

$$\frac{d\gamma}{dt} = \boldsymbol{\omega}_0 \cdot \boldsymbol{\beta} \tag{9.105}$$

and

$$\frac{d}{dt}(\boldsymbol{\beta}\gamma) = \boldsymbol{\omega}_0. \tag{9.106}$$

(b) Since the particle starts from rest show that we have

$$\boldsymbol{\beta}\gamma = \boldsymbol{\omega}_0 t. \tag{9.107}$$

For our configuration this implies

$$\beta_x = 0, \tag{9.108a}$$
$$\beta_y \gamma = \omega_0 t, \tag{9.108b}$$
$$\beta_z = 0. \tag{9.108c}$$

Further, deduce

$$\beta_y = \frac{\omega_0 t}{\sqrt{1+\omega_0^2 t^2}}. \tag{9.109}$$

Integrate again and use the initial condition to show that the motion is described by

$$y - y_0 = \frac{c}{\bar{\omega}_0}\left[\sqrt{1+\bar{\omega}_0^2 t^2} - 1\right]. \tag{9.110}$$

Rewrite the solution in the form

$$\left(y - y_0 + \frac{c}{\omega_0}\right)^2 - c^2 t^2 = \frac{c^2}{\omega_0^2}. \tag{9.111}$$

This represents a hyperbola passing through $y = y_0$ at $t = 0$. If we choose the initial position $y_0 = c/\omega_0$ we have

$$y^2 - c^2 t^2 = y_0^2. \tag{9.112}$$

(c) The (constant) proper acceleration associated with this motion is

$$\alpha = \omega_0 c = \frac{c^2}{y_0}. \tag{9.113}$$

A Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$y - y_0 = \frac{1}{2}\alpha t^2. \tag{9.114}$$

Show that the hyperbolic curve

$$y = y_0\sqrt{1+\frac{c^2 t^2}{y_0^2}} \tag{9.115}$$

in regions that satisfy

$$\omega_0 t \ll 1 \tag{9.116}$$

is approximately the parabolic curve

$$y = y_0 + \frac{1}{2}\alpha t^2 + \ldots. \tag{9.117}$$

2. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration α is described by equation of a hyperbola

$$z^2 - c^2 t^2 = z_0^2, \qquad z_0 = \frac{c^2}{\alpha}. \tag{9.118}$$

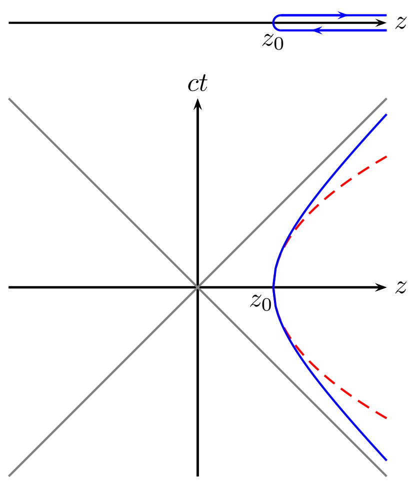

Figure 9.4: Problem 2

(a) This represents the world-line of a particle thrown from $z > z_0$ at $t < 0$ towards $z = z_0$ in region of constant (proper) acceleration α as described by the bold (blue) curve in the space-time diagram in Figure 2. In contrast a Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$z - z_0 = \frac{1}{2}\alpha t^2 \tag{9.119}$$

as described by the dashed (red) curve in the space-time diagram in Figure 2. Show that the hyperbolic curve

$$z = z_0\sqrt{1 + \frac{c^2t^2}{z_0^2}} \tag{9.120}$$

in regions that satisfy

$$t \ll \frac{c}{\alpha} \tag{9.121}$$

is approximately the parabolic curve

$$z = z_0 + \frac{1}{2}\alpha t^2 + \ldots . \tag{9.122}$$

(b) Recognize that the proper acceleration α does not have an upper bound.

(c) A large acceleration is achieved by taking an above turn while moving very fast. Thus, turning around while moving close to the speed of light c should achieve the highest acceleration. Show that $\alpha \to \infty$ corresponding to $z_0 \to 0$ represents this scenario. What is the equation of motion of a particle moving with infinite proper acceleration. To gain insight, plot world-lines of particles moving with $\alpha = c^2/z_0$, $\alpha = 10c^2/z_0$, and $\alpha = 100c^2/z_0$.

3. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration α is described by the equation of a hyperbola

$$z^2 - c^2t^2 = z_0^2, \qquad z_0 = \frac{c^2}{\alpha}. \tag{9.123}$$

This is the motion of a particle 'dropped' from $z = z_0$ at $t = 0$ in region of constant (proper) acceleration. See Figure 3. Using geometric (diagrammatic) arguments might be easiest to answer the following.

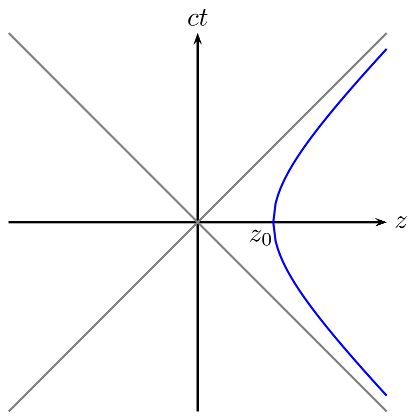

Figure 9.5: Problem 3

(a) Will a photon dispatched to 'chase' this particle at $t = 0$ from $z = 0$ ever catch up with it? If yes, when and where does it catch up?

(b) Will a photon dispatched to 'chase' this particle at $t = 0$ from $0 < z < z_0$ ever catch up with it? If yes, when and where does it catch up?

(c) Will a photon dispatched to 'chase' this particle, at $t = 0$ from $z < 0$ ever catch up with it? If yes, when and where does it catch up?

What are the implications for the observable part of our universe from this analysis?

4. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration g is described by the equation of a hyperbola

$$z_q(t) = \sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{9.124}$$

This is the motion of a particle that comes to existance at $z_q = +\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = z_0$, and then returns back to infinity. Assume you are positioned at the origin. If the particle is a source of light (imagine a flash light) at what time will the light first reach you at the origin? Where is the particle when this happens?

5. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration g is described by the equation of a hyperbola

$$z_2(t) = \sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{9.125}$$

This is the motion of a particle that comes to existance at $z_2 = +\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_2(0) = 0$ and $y_2(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = z_0$, and then returns back to infinity. Another particle is at rest at z_1

$$z_1(t) = z_1, \tag{9.126}$$

such that $0 < z_1 < z_0$. Assume that both particles emit photons continuously.

(a) At what time do photons emitted by 2 first reach 1? Where is particle 2 when this happens?

(b) At what time is the last photon that reaches 2 emitted by 1? Where is particle 2 when this happens?

(c) Do all the photons emitted by 1 reach 2?

(d) Do all the photons emitted by 2 reach 1?

6. (**20 points.**) The path of a relativistic particle 1 moving along a straight line with constant (proper) acceleration g is described by the equation of a hyperbola

$$z_1(t) = \sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{9.127}$$

This is the motion of a particle that comes to existance at $z_1 = +\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = z_0$, and then returns back to infinity. Consider another relavistic particle 2 undergoing hyperbolic motion given by

$$z_2(t) = -\sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{9.128}$$

This is the motion of a particle that comes to existance at $z_2 = -\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = -z_0$, and then returns back to negative infinity. The world-line of particle 1 is the blue curve in Figure 6, and the world-line of particle 2 is the red curve in Figure 6. Using geometric (diagrammatic) arguments might be easiest to answer the following. Imagine the particles are sources of light (imagine a flash light pointing towards origin).

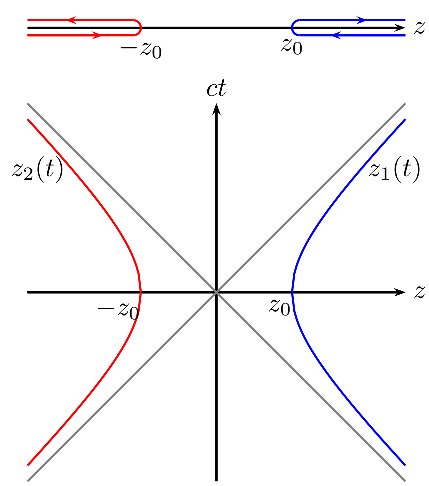

Figure 9.6: Problem 6

(a) At what time will the light from particle 1 first reach particle 2? Where are the particles when this happens?

(b) At what time will the light from particle 2 first reach particle 1? Where are the particles when this happens?

(c) Can the particles communicate with each other?

(d) Can the particles ever detect the presence of the other? In other words, can one particle be aware of the existence of the other? What can you deduce about the observable part of our universe from this analysis?

7. (**20 points.**) Two masses (one heavier than the other) move with constant proper acceleration α, after they are dropped from position $x_0 = c^2/\alpha$. Does the time taken to fall a given distance depend on mass?

Recall that Aristotle (384-322 BC) presumed that the time taken to fall a given distance depended on mass. Galileo (1564-1642) argued, based on a famous thought experiment (refer Wikipedia) that the time taken to fall a given distance is independent of mass.

(a) Consider an electron and a proton connected by a hypothetical string. What is the tension in the string when they move in a uniform electric field (which leads to proper acceleration). We will have to dictate how the distance between them changes.

(b) What about charges of different masses in an electric field?

(c) What about a hydrogen atom? How does electrostatic energy associated to the hydrogen atom fall?

(d) Do these considerations involve a Poincare stress?

Keywords: Trouton-Noble experiment, Laue current, 4/3 problem.
NOTE: This problem needs thought and scrutiny!

9.6.3 Charge particle in a uniform electric field with an initial velocity normal to electric field: Hyperbolic motion

1. (**20 points.**) A relativisitic particle in a uniform electric field is described by the equations

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v}, \tag{9.129a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{9.129b}$$

where

$$E = mc^2\gamma, \tag{9.130a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{9.130b}$$

and

$$\mathbf{F} = q\mathbf{E}. \tag{9.131}$$

Let us consider the configuration with the electric field in the $\hat{\mathbf{y}}$ direction,

$$\mathbf{E} = E\,\hat{\mathbf{y}}, \tag{9.132}$$

and initial conditions

$$\mathbf{v}(0) = v_0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{9.133a}$$

$$\mathbf{x}(0) = x_0\,\hat{\mathbf{x}} + y_0\,\hat{\mathbf{y}} + z_0\,\hat{\mathbf{z}}. \tag{9.133b}$$

We will use the associated definitions $\boldsymbol{\beta}_0 = \mathbf{v}(0)/c$ and $\boldsymbol{\gamma}_0 = 1/\sqrt{1-\beta_0^2}$.

(a) In terms of the definition

$$\boldsymbol{\alpha} = \boldsymbol{\omega}_0 c = \frac{q\mathbf{E}}{m}, \tag{9.134}$$

show that the equations of motion are given by

$$\frac{d\gamma}{dt} = \boldsymbol{\omega}_0 \cdot \boldsymbol{\beta} \tag{9.135}$$

and

$$\frac{d}{dt}(\boldsymbol{\beta}\gamma) = \boldsymbol{\omega}_0. \tag{9.136}$$

(b) For our configuration show that

$$\boldsymbol{\beta}\gamma = \boldsymbol{\omega}_0 t + \beta_0\gamma_0\hat{\mathbf{x}}, \tag{9.137}$$

such that

$$\beta_x\gamma = \beta_0\gamma_0, \tag{9.138a}$$
$$\beta_y\gamma = \omega_0 t, \tag{9.138b}$$
$$\beta_z\gamma = 0. \tag{9.138c}$$

Using $\beta_z\gamma = 0$, learn that

$$\frac{\beta_z^2}{1-\beta_x^2-\beta_y^2-\beta_z^2} = 0 \tag{9.139}$$

and in conjunction with $\beta_x\gamma = \beta_0\gamma_0$ deduce that

$$\beta_z = 0 \tag{9.140}$$

and

$$\frac{\beta_x^2}{\beta_0^2} + \beta_y^2 = 1. \tag{9.141}$$

Thus, deduce

$$\gamma^2 = \omega_0^2 t^2 + \gamma_0^2 \tag{9.142}$$

and

$$\beta_x^2 + \beta_y^2 = \beta_0^2 + \frac{\beta_y^2}{\gamma_0^2}. \tag{9.143}$$

Further, deduce

$$\beta_y = \frac{\bar{\omega}_0 t}{\sqrt{1+\bar{\omega}_0^2 t^2}} \tag{9.144}$$

and

$$\beta_x = \frac{\beta_0}{\sqrt{1+\bar{\omega}_0^2 t^2}}, \tag{9.145}$$

where

$$\bar{\omega}_0 = \frac{\omega_0}{\gamma_0}. \tag{9.146}$$

Integrate again and use the initial condition to show that the motion is described by

$$x - x_0 = \frac{v_0}{\bar{\omega}_0}\sinh^{-1}\bar{\omega}_0 t, \tag{9.147a}$$
$$y - y_0 = \frac{c}{\bar{\omega}_0}\left[\sqrt{1+\bar{\omega}_0^2 t^2} - 1\right], \tag{9.147b}$$
$$z - z_0 = 0. \tag{9.147c}$$

(c) Show that for $v_0 = 0$ we reproduce the solution for a particle starting from rest. Next, for

$$\bar{\omega}_0 t \ll 1 \tag{9.148}$$

and

$$\alpha = \bar{\omega}_0 c \tag{9.149}$$

obtain the non-relativistic limits,

$$x - x_0 = v_0 t, \tag{9.150a}$$
$$y - y_0 = \frac{1}{2}\alpha t^2, \tag{9.150b}$$
$$z - z_0 = 0. \tag{9.150c}$$

Hint: Recall the series expansion

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) = x + \dots. \tag{9.151}$$

(d) For the choice of initial position,

$$x_0 = 0, \qquad y_0 = \frac{c}{\bar{\omega}_0} = \frac{c^2\gamma_0}{\alpha}, \qquad z_0 = 0, \tag{9.152}$$

show that the trajectory is a catenary,

$$y = y_0 \cosh\left(\frac{\bar{\omega}_0}{v_0} x\right). \tag{9.153}$$

9.7 References

[1] J. Kocik. *An interactive applet for exploring relativistic velocity addition.* Link to Webpage.

[2] J. Kocik. "Geometric diagram for relativistic addition of velocities". In: *Am. J. Phys.* 80 (Aug. 2012), pp. 737–739. DOI: 10.1119/1.4730931. arXiv: 1408.2435.

Chapter 10

Lorentz covariance of electrodynamic quantities

10.1 Maxwell equations

1. In terms of the four-vector potential
$$cA^\mu = (\phi, c\mathbf{A}) \tag{10.1}$$
the Maxwell field tensor $F_{\mu\nu}$ is defined as
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{10.2}$$
Note that, by construction, the field tensor is antisymmetric. Recall,
$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right). \tag{10.3}$$
Using the expression for the electric and magnetic field in terms of the potentials,
$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial}{\partial t}\mathbf{A}, \tag{10.4a}$$
$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{10.4b}$$
in Eq. (10.2), recognize
$$cF_{0i} = -E_i \tag{10.5}$$
and
$$F_{ij} = \varepsilon_{ijk} B^k. \tag{10.6}$$
The tensor structure is more explicitly visualized in the form
$$cF_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & cB_3 & -cB_2 \\ E_2 & -cB_3 & 0 & cB_1 \\ E_3 & cB_2 & -cB_1 & 0 \end{pmatrix}. \tag{10.7}$$

2. In terms of the four-current
$$j^\mu = (c\rho, \mathbf{j}), \tag{10.8}$$
show that the inhomogeneous Maxwell equations,
$$\boldsymbol{\nabla} \cdot \varepsilon_0 \mathbf{E} = \rho, \tag{10.9a}$$
$$\boldsymbol{\nabla} \times \mathbf{H} - \frac{\partial}{\partial t}\varepsilon_0 \mathbf{E} = \mathbf{j}, \tag{10.9b}$$

where $\mathbf{B} = \mu_0 \mathbf{H}$, are summarized in the covariant equation

$$\partial_\beta F^{\alpha\beta} = \mu_0 j^\alpha. \tag{10.10}$$

3. Show that

$$\partial_\alpha \partial_\beta F^{\alpha\beta} = 0, \tag{10.11}$$

using the antisymmetry of the field tensor. Thus, derive

$$\partial_\alpha j^\alpha = 0, \tag{10.12}$$

and recognize it as the statement of conservation of charge,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0, \tag{10.13}$$

in covariant form.

4. The dual Maxwell field tensor is defined as

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}, \tag{10.14}$$

where the total antisymmetrical tensor of the fourth rank is normalized to

$$\varepsilon^{0123} = +1. \tag{10.15}$$

Show that

$$\tilde{F}_{0i} = -B_i \tag{10.16}$$

and

$$\tilde{F}_{ij} = -\varepsilon_{ijk} E^k. \tag{10.17}$$

The dual field tensor is more explicitly visualized in the form

$$c\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & -cB_1 & -cB_2 & -cB_3 \\ cB_1 & 0 & -E_3 & E_2 \\ cB_2 & E_3 & 0 & -E_1 \\ cB_3 & -E_2 & E_1 & 0 \end{pmatrix}. \tag{10.18}$$

Using antisymmetry derive

$$\partial_\beta \tilde{F}^{\alpha\beta} = 0 \tag{10.19}$$

and show that it summarizes the homogeneous Maxwell equations,

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{10.20a}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{10.20b}$$

in covariant form.

5. Show that

$$-j^\alpha A_\alpha = \rho\phi - \mathbf{j} \cdot \mathbf{A}, \tag{10.21}$$

and recognize this as the electrodynamic interaction energy in covariant form.

6. In terms of the four-velocity

$$u^\alpha = (c\gamma, \mathbf{v}\gamma) \tag{10.22}$$

and four-momentum

$$p^\alpha = (\frac{E}{c}, \mathbf{p}) = mu^\alpha \tag{10.23}$$

show that the covariant Lorentz force equation is

$$\frac{dp^\alpha}{ds} = qF^{\alpha\beta}u_\beta. \tag{10.24}$$

In particular, show that

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v}, \tag{10.25a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{10.25b}$$

where

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}. \tag{10.26}$$

10.2 Conservation equations

Show that

$$F^{\mu\nu}j_\nu + \partial_\nu t^{\mu\nu} = 0. \tag{10.27}$$

Identify the energy-momentum stress tensor

$$t^{\mu\nu} = F^{\mu\lambda}F^\nu{}_\lambda + g^{\mu\nu}\mathcal{L}, \tag{10.28}$$

where

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{10.29}$$

10.3 Lorentz invariant constructions

1. (**20 points.**) In terms of the four-vector potential

$$cA^\mu = (\phi, c\mathbf{A}) \tag{10.30}$$

the Maxwell field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{10.31}$$

and the corresponding dual tensor is defined as

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}. \tag{10.32}$$

Derive the following relations, which involve quantities that remain invariant under Lorentz transformations.

$$c^2F^{\mu\nu}F_{\mu\nu} = 2(c^2B^2 - E^2). \tag{10.33a}$$

$$c^2\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = -2(c^2B^2 - E^2). \tag{10.33b}$$

$$c^2F^{\mu\nu}\tilde{F}_{\mu\nu} = 4\,c\mathbf{B}\cdot\mathbf{E}. \tag{10.33c}$$

2. (**20 points.**) Eigenvalues of the energy momentum tensor. (If we choose $c = 1$, which is easily undone by replacing $\mathbf{E} \to \frac{1}{c}\mathbf{E}$ everywhere.)

   (a) Using

$$cF_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & cB_3 & -cB_2 \\ E_2 & -cB_3 & 0 & cB_1 \\ E_3 & cB_2 & -cB_1 & 0 \end{pmatrix}. \tag{10.34}$$

   and

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{10.35}$$

   derive

$$cF^{\mu\lambda}c\tilde{F}_{\lambda\nu} = \delta^\mu{}_\nu \mathbf{E}\cdot c\mathbf{B}, \tag{10.36a}$$

$$c\tilde{F}^{\mu\lambda}c\tilde{F}_{\lambda\nu} - cF^{\mu\lambda}cF_{\lambda\nu} = \delta^\mu{}_\nu (c^2B^2 - E^2). \tag{10.36b}$$

   (b) Define

$$\mathcal{L} = \frac{\varepsilon_0 E^2}{2} - \frac{B^2}{2\mu_0} \quad \text{and} \quad \mathcal{G} = \varepsilon_0 \mathbf{E}\cdot c\mathbf{B}, \tag{10.37}$$

   such that

$$-2\mu_0 c^2 \mathcal{L} = c^2B^2 - E^2 \quad \text{and} \quad \mu_0 c^2 \mathcal{G} = \mathbf{E}\cdot c\mathbf{B}. \tag{10.38}$$

   Thus, construct matrix (or dyadic) equations

$$\mathbf{F}\cdot\tilde{\mathbf{F}} = \mu_0 \mathcal{G}\mathbf{1}, \tag{10.39a}$$

$$\tilde{\mathbf{F}}\cdot\tilde{\mathbf{F}} - \mathbf{F}\cdot\mathbf{F} = -2\mu_0\mathcal{L}\mathbf{1}, \tag{10.39b}$$

   in terms of matrices (or dyadics) $\mathbf{F}$ and $\tilde{\mathbf{F}}$.

   (c) Show that the eigenvalues λ of the field tensor $\mathbf{F}/\sqrt{\mu_0}$ satisfy the quartic equation

$$\lambda^4 - 2\mathcal{L}\lambda^2 - \mathcal{G}^2 = 0. \tag{10.40}$$

   (d) Evaluate the eigenvalues to be $\pm\lambda_1$ and $\pm\lambda_2$ where

$$\lambda_1 = \sqrt{\mathcal{L} - \sqrt{\mathcal{L}^2 + \mathcal{G}^2}}, \tag{10.41a}$$

$$\lambda_2 = \sqrt{\mathcal{L} + \sqrt{\mathcal{L}^2 + \mathcal{G}^2}}. \tag{10.41b}$$

3. (**20 points.**) The eigenvalues λ of the field tensor $F^{\mu\nu}/\sqrt{\mu_0}$ satisfy the quartic equation

$$\lambda^4 - 2\mathcal{L}\lambda^2 - \mathcal{G}^2 = 0 \tag{10.42}$$

   in terms of

$$\mathcal{L} = \frac{\varepsilon_0 E^2}{2} - \frac{B^2}{2\mu_0} \quad \text{and} \quad \mathcal{G} = \varepsilon_0 \mathbf{E}\cdot c\mathbf{B}, \tag{10.43}$$

   such that

$$-2\mu_0 c^2 \mathcal{L} = c^2B^2 - E^2 \quad \text{and} \quad \mu_0 c^2 \mathcal{G} = \mathbf{E}\cdot c\mathbf{B}. \tag{10.44}$$

   (a) Evaluate the eigenvalues to be $\pm\lambda_1$ and $\pm\lambda_2$ where

$$\lambda_1 = \sqrt{\mathcal{L} - \sqrt{\mathcal{L}^2 + \mathcal{G}^2}}, \tag{10.45a}$$

$$\lambda_2 = \sqrt{\mathcal{L} + \sqrt{\mathcal{L}^2 + \mathcal{G}^2}}. \tag{10.45b}$$

(b) In terms of the complex field

$$c\mathbf{X} = \frac{\mathbf{E} + ic\mathbf{B}}{\sqrt{2}} \tag{10.46}$$

show that

$$\mathcal{Z} = \frac{1}{\mu_0}\mathbf{X}\cdot\mathbf{X} = \mathcal{L} + i\mathcal{G} \tag{10.47}$$

and

$$\mathcal{Z}^* = \mathcal{L} - i\mathcal{G}. \tag{10.48}$$

Then, express the eigenvalues as

$$\frac{\lambda}{\sqrt{\mu_0}} = \pm\frac{1}{\sqrt{2}}\left(\sqrt{\mathcal{L}+i\mathcal{G}} \pm \sqrt{\mathcal{L}-i\mathcal{G}}\right). \tag{10.49}$$

Hint: Substitute $\mathcal{Z} = Re^{i\theta}$.

(c) Show that
   i. if $c^2B^2 - E^2 = 0$, then the eigenvalues are $\pm\sqrt{\mathcal{G}}$ and $\pm i\sqrt{\mathcal{G}}$.
   ii. if $\mathbf{E}\cdot c\mathbf{B} = 0$, then the eigenvalues are 0, 0, and $\pm\sqrt{2\mathcal{L}}$.

(d) Is the following true?
   i. There is no Lorentz transformation connecting two reference frames such that the field is purely magnetic in origin in one and purely electric in origin in the other.
   ii. If $c^2B^2 - E^2 > 0$ in a frame, then there exists a frame in which the field is purely magnetic.
   iii. If $c^2B^2 - E^2 < 0$ in a frame, then there exists a frame in which the field is purely electric.
   iv. If $c^2B^2 - E^2 = 0$ in a frame, then it is so in every frame.
   v. $\mathbf{E}\cdot c\mathbf{B} > 0$ in a frame, then there exists a frame in which the fields are parallel.
   vi. $\mathbf{E}\cdot c\mathbf{B} < 0$ in a frame, then there exists a frame in which the fields are antiparallel.
   vii. $\mathbf{E}\cdot c\mathbf{B} = 0$ in a frame, then it is so in every frame.
   viii. An electromagnetic plane wave is characterized by $c^2B^2 - E^2 = 0$ and $\mathbf{E}\cdot c\mathbf{B} = 0$.

4. (**40 points.**) The electric and magnetic fields transform under a Lorentz transformation (for boost in z direction) as

$$E'_x(\mathbf{r}',t') = \gamma\, E_x(\mathbf{r},t) + \beta\gamma\, cB_y(\mathbf{r},t), \tag{10.50a}$$
$$cB'_y(\mathbf{r}',t') = \beta\gamma\, E_x(\mathbf{r},t) + \gamma\, cB_y(\mathbf{r},t), \tag{10.50b}$$
$$E'_z(\mathbf{r}',t') = E_z(\mathbf{r},t) \tag{10.50c}$$

$$cB'_x(\mathbf{r}',t') = \gamma\, cB_x(\mathbf{r},t) - \beta\gamma\, E_y(\mathbf{r},t), \tag{10.51a}$$
$$E'_y(\mathbf{r}',t') = -\beta\gamma\, cB_x(\mathbf{r},t) + \gamma\, E_y(\mathbf{r},t), \tag{10.51b}$$
$$cB'_z(\mathbf{r}',t') = cB_z(\mathbf{r},t), \tag{10.51c}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. The transformed values of the coordinates and the fields are distinguished by a prime. Derive the invariance properties

$$\mathbf{E}'(\mathbf{r}',t')\cdot\mathbf{B}'(\mathbf{r}',t') = \mathbf{E}(\mathbf{r},t)\cdot\mathbf{B}(\mathbf{r},t) \tag{10.52}$$

and

$$\mathbf{E}'(\mathbf{r}',t')^2 - c^2\mathbf{B}'(\mathbf{r}',t')^2 = \mathbf{E}(\mathbf{r},t)^2 - c^2\mathbf{B}(\mathbf{r},t)^2. \tag{10.53}$$

5. (**20 points.**) Let an infinitely thin plate occupying the $y = 0$ plane consist of a uniform charge density flowing in the $\hat{\mathbf{x}}$ direction described by drift velocity $\beta_d = v/c$.

(a) Show that the electric and magnetic field for this configuration is given by

$$\mathbf{E} = \eta(y)\,\hat{\mathbf{y}}\,\frac{\sigma}{2\varepsilon_0}, \tag{10.54a}$$
$$c\mathbf{B} = \eta(y)\,\hat{\mathbf{z}}\,\beta_d E, \tag{10.54b}$$

where

$$\eta(y) = \begin{cases} 1, & y > 0, \\ -1, & y < 0. \end{cases} \tag{10.55}$$

Thus, we have

$$cB = \beta_d E. \tag{10.56}$$

Recall that the motion of a point charge in this field configuration is a cycloid,

$$x(t) - v_q t = R \sin \omega_c t, \tag{10.57a}$$

$$y(t) - R = R \cos \omega_c t, \tag{10.57b}$$

that satisfies

$$\big[x(t) - v_q t\big]^2 + \big[y(t) - R\big]^2 = R^2, \tag{10.58}$$

where

$$\omega_c = \frac{qB}{m}, \quad v_q = \frac{E}{B} \quad \text{and} \quad R = \frac{v_q}{\omega_c}. \tag{10.59}$$

(b) Show that under a Lorentz transformation (for boost in x direction) the electric and magnetic fields transform as

$$\mathbf{E}' = \hat{\mathbf{y}}\, E', \tag{10.60a}$$

$$c\mathbf{B}' = \hat{\mathbf{z}}\, B'\eta(y), \tag{10.60b}$$

where

$$E' = \gamma(E - \beta cB), \tag{10.61a}$$

$$cB' = \gamma(cB - \beta E). \tag{10.61b}$$

Verify that

$$E'^2 - (cB)'^2 = E^2 - (cB)^2 \tag{10.62}$$

and

$$\mathbf{E}' \cdot \mathbf{B}' = \mathbf{E} \cdot \mathbf{B} = 0. \tag{10.63}$$

(c) Verify that for $\beta = \beta_d < 1$ we have $B' = 0$ and $E' = E/\gamma_d$. Investigate what happens to the radius R and the pitch of the cycloid $2\pi R$ in this case.

(d) Note that for $\beta = E/(cB) > 1$ we have $B' = B/\gamma$ and $E' = 0$. Investigate what happens.