(Preview of) Midterm Exam No. 02 (2026 Spring)

PHYS 520B: ELECTROMAGNETIC THEORY

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1. (**20 points, Bring from home.**) A current I flows in an infinitely long conducting wire placed on the z-axis such that there exists a (rest) frame with respect to which the charge density is zero everywhere. That is,

$$\rho(\mathbf{r},t) = 0, \tag{1a}$$
$$\mathbf{j}(\mathbf{r},t) = \hat{\mathbf{k}}\, I\, \delta(x)\delta(y). \tag{1b}$$

Verify that the charge conservation equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0, \tag{2}$$

is satisfied by these densities.

(a) In a frame moving with velocity v, $\beta = v/c$, parallel to the z axis, the charge and current densities are given by

$$\begin{bmatrix} c\rho'(\mathbf{r}',t') \\ j_1'(\mathbf{r}',t') \\ j_2'(\mathbf{r}',t') \\ j_3'(\mathbf{r}',t') \end{bmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ I\,\delta(x)\delta(y) \end{bmatrix} \tag{3}$$

and derive

$$c\rho'(\mathbf{r}',t') = \beta\gamma I\,\delta(x')\delta(y'), \tag{4a}$$
$$j_1'(\mathbf{r}',t') = 0, \tag{4b}$$
$$j_2'(\mathbf{r}',t') = 0, \tag{4c}$$
$$j_3'(\mathbf{r}',t') = \gamma I\,\delta(x')\delta(y'). \tag{4d}$$

Verify that the charge conservation equation in this frame,

$$\frac{\partial}{\partial t'}\rho'(\mathbf{r}',t') + \boldsymbol{\nabla}'\cdot\mathbf{j}'(\mathbf{r}',t') = 0, \tag{5}$$

is satisfied by these densities.

(b) Show that the electric scalar potential and the magnetic vector potential in the rest frame are given by

$$\phi(\mathbf{r},t) = 0, \tag{6a}$$

$$c\mathbf{A}(\mathbf{r},t) = -\hat{\mathbf{k}}\left(\frac{\mu_0 c}{4\pi}\right) 2I \ln\frac{\rho}{L_z}. \tag{6b}$$

In the lab frame, starting from Maxwell equations, without using special relativity, derive the expressions

$$\phi'(\mathbf{r}',t') = -\beta\gamma\left(\frac{\mu_0 c}{4\pi}\right) 2I \ln\frac{\rho'}{2L_z}, \tag{7a}$$

$$c\mathbf{A}'(\mathbf{r}',t') = -\hat{\mathbf{k}}\gamma\left(\frac{\mu_0 c}{4\pi}\right) 2I \ln\frac{\rho'}{2L_z}. \tag{7b}$$

Show that the scalar and vector potentials together transform like a four-vector,

$$\begin{bmatrix} \phi'(\mathbf{r}',t') \\ cA'_1(\mathbf{r}',t') \\ cA'_2(\mathbf{r}',t') \\ cA'_3(\mathbf{r}',t') \end{bmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ cA_3(\mathbf{r},t) \end{bmatrix}. \tag{8}$$

(c) Without using special relativity, show that the electric and magnetic field in the rest frame are given by

$$\mathbf{E}(\mathbf{r},t) = 0, \tag{9a}$$

$$c\mathbf{B}(\mathbf{r},t) = \left(\frac{\mu_0 c}{4\pi}\right) 2I \frac{\left[-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}\right]}{[x^2+y^2]}, \tag{9b}$$

and in the lab frame are given by

$$\mathbf{E}'(\mathbf{r}',t') = \beta\gamma\left(\frac{\mu_0 c}{4\pi}\right) 2I \frac{\left[x'\hat{\mathbf{i}} + y'\hat{\mathbf{j}}\right]}{\left[x'^2+y'^2\right]}, \tag{10a}$$

$$c\mathbf{B}'(\mathbf{r}',t') = \gamma\left(\frac{\mu_0 c}{4\pi}\right) 2I \frac{\left[-y'\hat{\mathbf{i}} + x'\hat{\mathbf{j}}\right]}{\left[x'^2+y'^2\right]}. \tag{10b}$$

Verify that the fields transform like a four-tensor,

$$\begin{bmatrix} 0 & E'_1(\mathbf{r}',t') & E'_2(\mathbf{r}',t') & E'_3(\mathbf{r}',t') \\ -E'_1(\mathbf{r}',t') & 0 & cB'_3(\mathbf{r}',t') & -cB'_2(\mathbf{r}',t') \\ -E'_2(\mathbf{r}',t') & -cB'_3(\mathbf{r}',t') & 0 & cB'_1(\mathbf{r}',t') \\ -E'_3(\mathbf{r}',t') & cB'_2(\mathbf{r}',t') & -cB'_1(\mathbf{r}',t') & 0 \end{bmatrix}$$
$$= \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -cB_2(\mathbf{r},t) \\ 0 & 0 & 0 & cB_1(\mathbf{r},t) \\ 0 & cB_2(\mathbf{r},t) & -cB_1(\mathbf{r},t) & 0 \end{bmatrix} \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}. \tag{11}$$

That is,

$$E_1'(\mathbf{r}',t') = +\beta\gamma c B_2(\mathbf{r},t), \qquad B_1'(\mathbf{r}',t') = \gamma B_1(\mathbf{r},t), \tag{12a}$$
$$E_2'(\mathbf{r}',t') = -\beta\gamma c B_1(\mathbf{r},t), \qquad B_2'(\mathbf{r}',t') = \gamma B_2(\mathbf{r},t), \tag{12b}$$
$$E_3'(\mathbf{r}',t') = 0, \qquad B_3'(\mathbf{r}',t') = 0. \tag{12c}$$

2. (**20 points, Bring from home.**) Consider the following charge distribution, for example, of a line charge,

$$\rho(\mathbf{r},t) = \lambda\delta(x)\delta(y), \tag{13a}$$
$$\mathbf{j}(\mathbf{r},t) = 0. \tag{13b}$$

Show that the current four-vector for this configuration is time-like. In a frame moving with velocity v, $\beta = v/c$, parallel to the z axis, the charge and current densities are given by

$$\begin{bmatrix} c\rho'(\mathbf{r}',t') \\ j_1'(\mathbf{r}',t') \\ j_2'(\mathbf{r}',t') \\ j_3'(\mathbf{r}',t') \end{bmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{bmatrix} c\lambda\delta(x)\delta(y) \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

and derive

$$\rho'(\mathbf{r}',t') = \gamma\,\lambda\,\delta(x')\delta(y'), \tag{15a}$$
$$j_1'(\mathbf{r}',t') = 0, \tag{15b}$$
$$j_2'(\mathbf{r}',t') = 0, \tag{15c}$$
$$j_3'(\mathbf{r}',t') = \beta\gamma\,\lambda\,\delta(x')\delta(y'). \tag{15d}$$

Next, consider the following charge distribution, for example, of a neutral conducting wire in which the positive charges are moving relative to the negative charges,

$$\rho(\mathbf{r},t) = 0, \tag{16a}$$
$$\mathbf{j}(\mathbf{r},t) = \hat{\mathbf{k}}\,I\,\delta(x)\delta(y). \tag{16b}$$

Show that the current four-vector for this configuration is space-like. Argue that the above configurations are physically distinct, because there exists no Lorentz transformation that can transform one to the other. Find the electric and magnetic field for these configurations and discuss why such transformations are not possible.

3. (**20 points, Bring from home.**) An infinitely thin neutral conducting plate is placed on the $z = 0$ plane such that there exists a (rest) frame with respect to which the charge and current densities are

$$\rho(\mathbf{r},t) = 0, \tag{17a}$$
$$\mathbf{j}(\mathbf{r},t) = \hat{\mathbf{y}}\,\lambda\,\delta(z), \tag{17b}$$

where $\lambda = I/L_x$ is current per unit length on the plate. Verify that the charge conservation equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0, \tag{18}$$

is satisfied by these densities.

(a) In a frame moving with velocity v, $\beta = v/c$, parallel to the z axis, the charge and current densities are given by

$$\begin{bmatrix} c\rho'(\mathbf{r}',t') \\ j'_1(\mathbf{r}',t') \\ j'_2(\mathbf{r}',t') \\ j'_3(\mathbf{r}',t') \end{bmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ \lambda\delta(z) \\ 0 \end{bmatrix} \tag{19}$$

and derive

$$\rho'(\mathbf{r}',t') = 0, \tag{20a}$$

$$j'_1(\mathbf{r}',t') = 0, \tag{20b}$$

$$j'_2(\mathbf{r}',t') = \frac{\lambda}{\gamma}\delta(z' - vt'), \tag{20c}$$

$$j'_3(\mathbf{r}',t') = 0. \tag{20d}$$

We used $|a|\delta(as) = \delta(s)$. Verify that the charge conservation equation in this frame,

$$\frac{\partial}{\partial t'}\rho'(\mathbf{r}',t') + \boldsymbol{\nabla}' \cdot \mathbf{j}'(\mathbf{r}',t') = 0, \tag{21}$$

is satisfied by these densities.

(b) Show that the electric scalar potential and the magnetic vector potential in the rest frame are given by

$$\phi(\mathbf{r},t) = 0, \tag{22a}$$

$$\mathbf{A}(\mathbf{r},t) = \hat{\mathbf{y}}\frac{\mu_0\lambda}{2}\Big[R - |z|\Big], \quad |z| \ll R. \tag{22b}$$

In the lab frame, starting from Maxwell equations, derive the expressions

$$\phi'(\mathbf{r}',t') = 0, \tag{23a}$$

$$\mathbf{A}'(\mathbf{r}',t') = \hat{\mathbf{y}}\frac{\mu_0\lambda}{2}\Big[R - \gamma|z' - vt'|\Big], \quad |z' - vt'| \ll R. \tag{23b}$$

Show that the scalar and vector potentials together transform like a four-vector,

$$\begin{bmatrix} \phi'(\mathbf{r}',t') \\ cA'_1(\mathbf{r}',t') \\ cA'_2(\mathbf{r}',t') \\ cA'_3(\mathbf{r}',t') \end{bmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{bmatrix} 0 \\ 0 \\ cA_2(\mathbf{r},t) \\ 0 \end{bmatrix}. \tag{24}$$

(c) The electric and magnetic field in the rest frame are given by

$$\mathbf{E}(\mathbf{r},t) = 0, \tag{25a}$$

$$c\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{x}}\frac{\mu_0\lambda}{2}\eta(z), \tag{25b}$$

where $\eta(z) = +1$, if $z > 0$, and $\eta(z) = -1$, if $z < 0$, and in the lab frame, starting from

$$\mathbf{E}'(\mathbf{r}',t') = -\boldsymbol{\nabla}'\phi'(\mathbf{r}',t') - \frac{\partial}{\partial t'}\mathbf{A}'(\mathbf{r}',t'), \tag{26a}$$

$$\mathbf{B}'(\mathbf{r}',t') = 0, \tag{26b}$$

are given by

$$\frac{1}{c}\mathbf{E}'(\mathbf{r}',t') = -\beta\gamma\,\hat{\mathbf{y}}\,\frac{\mu_0\lambda}{2}\eta(z'-vt'), \tag{27a}$$

$$\mathbf{B}'(\mathbf{r}',t') = \gamma\,\hat{\mathbf{x}}\,\frac{\mu_0\lambda}{2}\eta(z'-vt'), \tag{27b}$$

$$\tag{27c}$$

Verify that the fields transform like a four-tensor,

$$\begin{bmatrix} 0 & E_1'(\mathbf{r}',t') & E_2'(\mathbf{r}',t') & E_3'(\mathbf{r}',t') \\ -E_1'(\mathbf{r}',t') & 0 & cB_3'(\mathbf{r}',t') & -cB_2'(\mathbf{r}',t') \\ -E_2'(\mathbf{r}',t') & -cB_3'(\mathbf{r}',t') & 0 & cB_1'(\mathbf{r}',t') \\ -E_3'(\mathbf{r}',t') & cB_2'(\mathbf{r}',t') & -cB_1'(\mathbf{r}',t') & 0 \end{bmatrix}$$

$$= \frac{\mu_0\lambda}{2}\eta(z)\begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}. \tag{28}$$

That is,

$$\frac{1}{c}E_1'(\mathbf{r}',t') = 0, \qquad B_1'(\mathbf{r}',t') = \gamma B_1(\mathbf{r},t), \tag{29a}$$

$$\frac{1}{c}E_2'(\mathbf{r}',t') = -\beta\gamma B_1(\mathbf{r},t), \qquad B_2'(\mathbf{r}',t') = 0, \tag{29b}$$

$$\frac{1}{c}E_3'(\mathbf{r}',t') = 0, \qquad B_3'(\mathbf{r}',t') = 0. \tag{29c}$$

We used

$$\eta(az) = \eta(z), \tag{30}$$

which can verified by showing that the derivative of both sides are the same,

$$\frac{\partial}{\partial z}\eta(az) = a\frac{\partial}{\partial az}\eta(az) = a\delta(az) = \delta(z). \tag{31}$$

4. (**20 points.**) Not available in preview mode.

5. (**20 points.**) Not available in preview mode.