Homework No. 03 (Spring 2026)

PHYS 520B: ELECTROMAGNETIC THEORY

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Due date: Thursday, 2026 Feb 05, 4.30pm

1. (**20 points.**) The Green function for a wave equation is

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}-\mathbf{r}', t-t') = \delta^{(3)}(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{1}$$

   (a) Let $\mathbf{r}' = 0$ and $t' = 0$. Then, Fourier transform in time to obtain

$$-\left(\nabla^2 + \frac{\omega^2}{c^2}\right) G(\mathbf{r};\omega) = \delta^{(3)}(\mathbf{r}), \tag{2}$$

   for a particular mode of frequency ω.

   (b) Integrate around the source at $\mathbf{r}'$ to obtain the continuity condition

$$\lim_{r\to 0}(4\pi r^2)\, \hat{\mathbf{r}} \cdot \boldsymbol{\nabla} G = -1. \tag{3}$$

   (c) Integrate the angular part, use spherical symmetry, to express the differential equation as

$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{\omega^2}{c^2}\right) G(r;\omega) = \frac{\delta(r)}{4\pi r^2} \tag{4}$$

   and rewrite the continuity condition in the form

$$\lim_{r\to 0} r^2 \frac{\partial G}{\partial r} = -\frac{1}{4\pi}. \tag{5}$$

   (d) In the static limit, $\omega \to 0$, the Green function reduces to

$$\lim_{\omega\to 0} G(r;\omega) = \frac{1}{4\pi r}. \tag{6}$$

   Thus, define $g(r;\omega)$ using

$$G(r;\omega) = \frac{g(r;\omega)}{4\pi r} \tag{7}$$

   and show that it satisfies the differential equation

$$-\left(\frac{d^2}{dr^2} + \frac{\omega^2}{c^2}\right) g(r;\omega) = \frac{\delta(r)}{r} \tag{8}$$

   with continuity condition

$$\lim_{r\to 0} g(r;\omega) = 1. \tag{9}$$

(e) Solve for $g(r;\omega)$ and find

$$g(r;\omega) = Ae^{i\frac{\omega}{c}r} + Be^{-i\frac{\omega}{c}r} \tag{10}$$

with the constraint

$$A + B = 1. \tag{11}$$

Thus, show that

$$G(\mathbf{r}-\mathbf{r}';\omega) = \frac{Ae^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{Be^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{12}$$

Fourier transform to show that

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{A\delta(t-t'-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|} + \frac{B\delta(t-t'+\frac{1}{c}|\mathbf{r}-\mathbf{r}'|)}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{13}$$

Requiring the Green function to be causal, that is, $t > t'$, show that $A = 1$ and $B = 0$.