Homework No. 05 (Spring 2026)

PHYS 520B: ELECTROMAGNETIC THEORY

Department of Physics, Southern Illinois University–Carbondale

Due date: Thursday, 2026 Feb 19, 4.30pm

0. Problems 3, 5, 6, and 7, are for submission.

1. (**20 points.**) Lorentz transformation describing a boost in the x-direction, y-direction, and z-direction, are

$$L_1 = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} \gamma_2 & 0 & -\beta_2\gamma_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_2\gamma_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} \gamma_3 & 0 & 0 & -\beta_3\gamma_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_3\gamma_3 & 0 & 0 & \gamma_3 \end{pmatrix}, \tag{1}$$

respectively. Transformation describing a rotation about the x-axis, y-axis, and z-axis, are

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\omega_1 & \sin\omega_1 \\ 0 & 0 & -\sin\omega_1 & \cos\omega_1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_2 & 0 & -\sin\omega_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\omega_2 & 0 & \cos\omega_2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_3 & \sin\omega_3 & 0 \\ 0 & -\sin\omega_3 & \cos\omega_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

respectively. For infinitesimal transformations, $\beta_i = \delta\beta_i$ and $\omega_i = \delta\omega_i$ use the approximations

$$\gamma_i \sim 1, \qquad \cos\omega_i \sim 1, \qquad \sin\omega_i \sim \delta\omega_i, \tag{3}$$

to identify the generator for boosts $\mathbf{N}$, and the generator for rotations the angular momentum $\mathbf{J}$,

$$\mathbf{L} = \mathbf{1} + \delta\boldsymbol{\beta}\cdot\mathbf{N} \qquad \text{and} \qquad \mathbf{R} = \mathbf{1} + \delta\boldsymbol{\omega}\cdot\mathbf{J}, \tag{4}$$

respectively. Then derive

$$\Big[N_1, N_2\Big] = N_1N_2 - N_2N_1 = J_3. \tag{5}$$

This states that boosts in perpendicular direction leads to rotation. (To gain insight of the statement, calculate $[J_1, J_2]$ and interpret the result.)

   (a) Is velocity addition commutative?

   (b) Is velocity addition associative?

   (c) Read a resource article (Wikipedia) on Wigner rotation.

2. (**20 points.**) The following are identities:

$$\text{Tr}A = A_i{}^i. \tag{6a}$$

$$\det A = \varepsilon_{i_1 i_2 \ldots i_n} A^{i_1}{}_1 A^{i_2}{}_2 \ldots A^{i_n}{}_n \tag{6b}$$

$$= \frac{1}{n!} \varepsilon_{i_1 i_2 \ldots i_n} \varepsilon^{i'_1 i'_2 \ldots i'_n} A^{i_1}{}_{i'_1} A^{i_2}{}_{i'_2} \ldots A^{i_n}{}_{i'_n}, \tag{6c}$$

where n is the dimension of the matrix A. Verify them for $n = 3$.

3. (**20 points.**) The Poincaré formula for the addition of (parallel) velocities is, $c = 1$,

$$v = \frac{v_a + v_b}{1 + v_a v_b}, \tag{7}$$

where v_a and v_b are velocities and c is speed of light in vacuum. Assuming that the Poincaré formula holds for all speeds, subluminal ($-1 < v_i < 1$), superluminal ($|v_i| > 1$), and speed of light, analyse what is obtained if you add a speed to an infinitely large superluminal speed, that is, $v_b \to \infty$. Hint: Inversion. Verify this using the interactive applet for exploring velocity addition at Kocik's web page [1]. Jerzy Kocik, from the department of Mathematics in SIUC, invented this geometric diagram that allows us to visualize the Poincaré formula. (Refer [2].)

4. (**30 points.**) Let

$$\tanh\theta = \beta, \tag{8}$$

where $\beta = v/c$. Addition of (parallel) velocities in terms of the parameter θ obeys the arithmatic addition

$$\theta = \theta_a + \theta_b. \tag{9}$$

   (a) Invert the expression in Eq. (8) to find the explicit form of θ in terms of β as a logarithm.

   (b) Show that Eq. (9) leads to the relation

$$\left(\frac{1+\beta}{1-\beta}\right) = \left(\frac{1+\beta_a}{1-\beta_a}\right)\left(\frac{1+\beta_b}{1-\beta_b}\right). \tag{10}$$

   (c) Using Eq. (10) derive the Poincaré formula for the addition of (parallel) velocities.

5. (**20 points.**) Lorentz transformation relates the energy E and momentum $\mathbf{p}$ of a particle when measured in different frames. For example, for the special case when the relative velocity and the velocity of the particle are parallel we have

$$\begin{pmatrix} E'/c \\ p' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E/c \\ p \end{pmatrix}. \tag{11}$$

Photons are massless spin 1 particles whose energy and momentum are $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$, such that $\omega = kc$. Thus, derive the relativistic Doppler effect formula

$$\omega' = \omega\sqrt{\frac{1+\beta}{1-\beta}}. \tag{12}$$

Contrast the above formula with the Doppler effect formula for sound.

6. (**20 points.**) Neutral π meson decays into two photons. That is,

$$\pi^0 \to \gamma_1 + \gamma_2. \tag{13}$$

Energy-momentum conservation for the decay in the laboratory frame, in which the meson is not necessarily at rest, is given by

$$p_\pi^\alpha = p_1^\alpha + p_2^\alpha. \tag{14}$$

Or, more specifically,

$$\left(\frac{E_\pi}{c}, \mathbf{p}\right) = \left(\frac{E_1}{c}, \mathbf{p}_1\right) + \left(\frac{E_2}{c}, \mathbf{p}_2\right), \tag{15}$$

where E_π and $\mathbf{p}$ are the energy and momentum of neutral π meson, and E_i's and $\mathbf{p}_i$'s are the energies and momentums of the photons. Thus, derive the relation

$$m_\pi^2 c^4 = 2E_1 E_2 (1 - \cos\theta), \tag{16}$$

where m_π is the mass of neutral π meson, and θ is the angle between the directions of $\mathbf{p}_1$ and $\mathbf{p}_2$.

7. (**20 points.**) Length contracts and time dilates. That is,

$$L = \frac{L_0}{\gamma}, \qquad T = T_0\gamma, \tag{17}$$

where L_0 and T_0 are proper length and proper time. Similarly, show that (for $\mathbf{v} \,||\, \mathbf{a}$)

$$|\mathbf{a}| = \frac{|\mathbf{a}_0|}{\gamma^3}, \tag{18}$$

where $|\mathbf{a}_0|$ is the proper acceleration measured in the instantaneaous rest frame of the particle. Further, for $\mathbf{v} \perp \mathbf{a}$ show that

$$|\mathbf{a}| = \frac{|\mathbf{a}_0|}{\gamma^2}. \tag{19}$$

References

[1] J. Kocik. An interactive applet for exploring relativistic velocity addition. http://www.mathoutlet.com/2016/08/relativistic-composition-of-velocities.html.

[2] J. Kocik. Geometric diagram for relativistic addition of velocities. *Am. J. Phys.*, 80:737–739, August 2012.