

Homework No. 10 (Spring 2026)

PHYS 520B: ELECTROMAGNETIC THEORY

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Due date: Thursday, 2026 Apr 23, 4.30pm

1. (20 points.) The magnetic field associated to radiation fields, in the frequency domain, is given by

$$c\mathbf{B}(\mathbf{r}, \omega) = -\hat{\mathbf{r}} \times \mathbf{F}(\theta, \phi; \omega) \frac{e^{ikr}}{r}, \quad (1)$$

where

$$\mathbf{F}(\theta, \phi; \omega) = \frac{\mu_0}{4\pi} (-i\omega) \mathbf{J}(\mathbf{k}, \omega), \quad (2)$$

where we have used the notation

$$\mathbf{k} = \frac{\omega}{c} \hat{\mathbf{r}} \quad (3)$$

for insight in the context of Fourier transformation. The associated electric field is given by

$$\mathbf{E}(\mathbf{r}, \omega) = -\hat{\mathbf{r}} \times c\mathbf{B}(\mathbf{r}, \omega), \quad (4)$$

and satisfies

$$c\mathbf{B}(\mathbf{r}, \omega) = \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r}, \omega). \quad (5)$$

The total energy E radiated into the solid angle $d\Omega$ per unit (positive, $0 \leq \omega < \infty$) frequency range $d\omega$ is given by

$$\frac{\partial}{\partial \omega} \frac{\partial E}{\partial \Omega} = \frac{1}{\pi} \frac{r^2}{c\mu_0} \left| c\mathbf{B}(\mathbf{r}, \omega) \right|^2. \quad (6)$$

- (a) Show that

$$\frac{\partial}{\partial \omega} \frac{\partial E}{\partial \Omega} = \frac{1}{4\pi} \left(\frac{\mu_0 c}{4\pi} \right) \frac{1}{\pi} \left| \frac{\omega}{c} \hat{\mathbf{r}} \times \mathbf{J}(\mathbf{r}, \omega) \right|^2. \quad (7)$$

Verify that $\omega J/c$ has the dimensions of charge. (Caution: J here is the Fourier transform of current density.) Thus, conclude that

$$\frac{\mu_0 c}{4\pi} = \frac{1}{4\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \quad (8)$$

has the dimensions of resistance. Quantum phenomena in electromagnetism is characterized by the Planck's constant h and the associated fine-structure constant

$$\alpha = \frac{1}{4\pi\epsilon_0} \frac{e^2}{\hbar c}, \quad (9)$$

a dimensionless physical constant. Verify that

$$\frac{\mu_0 c}{4\pi} = \frac{1}{4\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} = \alpha \frac{\hbar}{e^2} = 29.9792458 \Omega. \quad (10)$$

- (b) A loop antenna consists of a circular infinitely thin conductor of radius a carrying a time-dependent current. Let the circular conductor be centered at the origin and placed on the x - y plane such that

$$\mathbf{J}(\mathbf{r}', t') = \hat{\phi}' I_0 \sin \omega_0 t' \delta(\rho' - a) \delta(z'), \quad (11)$$

where $\rho' = \sqrt{x'^2 + y'^2}$ and $\hat{\phi}' = -\hat{x} \sin \phi' + \hat{y} \cos \phi'$. Evaluate the Fourier transform of the current density using

$$\mathbf{J}(\mathbf{k}, \omega) = \int d^3 r' \int dt' e^{-i\mathbf{k}\cdot\mathbf{r}'} e^{i\omega t'} \mathbf{J}(\mathbf{r}', t') \quad (12)$$

and show that

$$\mathbf{J}(\mathbf{k}, \omega) = \hat{\phi} 2\pi^2 a I_0 \delta(\omega - \omega_0) J_1(ka \sin \theta), \quad (13)$$

where $J_n(x)$ is the Bessel function of first kind.

Hint: You are expected to encounter the following integral

$$\int_0^{2\pi} d\phi' e^{-ika \sin \theta \cos(\phi - \phi')} [-\hat{x} \sin \phi' + \hat{y} \cos \phi']. \quad (14)$$

Substitute $\phi' - \phi = \phi''$ to obtain

$$\hat{\phi} \int_0^{2\pi} d\phi'' \cos \phi'' e^{-ika \sin \theta \cos \phi''} - \hat{\rho} \int_0^{2\pi} d\phi'' \sin \phi'' e^{-ika \sin \theta \cos \phi''}. \quad (15)$$

Use the integrals

$$\int_0^{2\pi} \frac{d\phi'}{2\pi} \cos \phi' e^{-ix \cos \phi'} = (-i) J_1(x) \quad (16)$$

and

$$\int_0^{2\pi} \frac{d\phi'}{2\pi} \sin \phi' e^{-ix \cos \phi'} = 0. \quad (17)$$

We also dropped the delta-function contribution associated to $\delta(\omega + \omega_0)$, because $0 \leq \omega < \infty$.

- (c) Show that

$$\frac{\partial}{\partial \omega} \frac{\partial P}{\partial \Omega} = P_0 \pi^2 (ka)^2 J_1^2(ka \sin \theta) \delta(\omega - \omega_0), \quad (18)$$

where

$$P_0 = \left(\frac{\mu_0 c}{4\pi} \right) I_0^2. \quad (19)$$

Here we used the interpretation

$$\delta(\omega - \omega_0) \delta(\omega - \omega_0) = \delta(\omega - \omega_0) \int_{-\infty}^{\infty} dt e^{i(\omega - \omega_0)t} = \delta(\omega - \omega_0) \int_{-\infty}^{\infty} dt = \delta(\omega - \omega_0) T, \quad (20)$$

where T is the infinite time for which the system is evolving. We used E/T to be the power P .

(d) Integration with respect to frequency yields the power radiated per unit solid angle

$$\frac{\partial P}{\partial \Omega} = P_0 \pi^2 (ka)^2 J_1^2(ka \sin \theta). \quad (21)$$

Plot the angular distribution of radiated power for $ka = 0.5, 2, 3, 4, 6$. Note that

$$ka = \frac{\omega_0}{c} a = 2\pi \frac{a}{\lambda_0}, \quad (22)$$

where λ_0 is the wavelength associated with the angular frequency ω_0 .