

Homework No. 12 (Spring 2026)

PHYS 520B: ELECTROMAGNETIC THEORY

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Due date: None

1. **(20 points.)** An electromagnetic scattering process is characterized by the (Sommerfeld) boundary conditions

$$\lim_{r \rightarrow \infty} \mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}_{\text{in}}(\mathbf{r}, \omega) + \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{1}) \cdot \mathbf{K}(\theta, \phi, \omega) \frac{e^{ikr}}{r}, \quad (1a)$$

$$\lim_{r \rightarrow \infty} c\mathbf{B}(\mathbf{r}, \omega) = c\mathbf{B}_{\text{in}}(\mathbf{r}, \omega) - (\hat{\mathbf{r}} \times \mathbf{1}) \cdot \mathbf{K}(\theta, \phi, \omega) \frac{e^{ikr}}{r}, \quad (1b)$$

where the scattering amplitude, up to a scale, is

$$\mathbf{K}(\theta, \phi, \omega) = -\frac{k^2}{4\pi} \int d^3r' e^{-i\mathbf{k}' \cdot \mathbf{r}'} \boldsymbol{\chi}(\mathbf{r}', \omega) \cdot \mathbf{E}(\mathbf{r}', \omega). \quad (2)$$

The electric field in Eq. (2) is determined by solving the dyadic differential equation

$$\left[\frac{c^2}{\omega^2} \nabla \times (\nabla \times \mathbf{1}) - \mathbf{1} \right] \cdot \mathbf{E}(\mathbf{r}, \omega) = \boldsymbol{\chi}(\mathbf{r}, \omega) \cdot \mathbf{E}(\mathbf{r}, \omega), \quad (3)$$

satisfying boundary conditions in Eq. (1). The incident wave is chosen to be a monochromatic plane wave of frequency ω in time domain

$$\mathbf{E}_{\text{in}}(\mathbf{r}, t) = \text{Re } \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad (4a)$$

$$\mathbf{B}_{\text{in}}(\mathbf{r}, t) = \text{Re } \mathbf{B}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad (4b)$$

that satisfies Maxwell equations, independently, which implies that $k = \frac{\omega}{c}$ and the direction of \mathbf{k} is constrained by with $\mathbf{k} \cdot \mathbf{E}_0 = 0$ and $\mathbf{k} \cdot \mathbf{B}_0 = 0$. In the frequency domain the incident field is given by

$$\mathbf{E}_0(\mathbf{r}, \omega) = \mathbf{E}_0 e^{i\mathbf{k} \cdot \mathbf{r}}, \quad (5a)$$

$$\mathbf{B}_0(\mathbf{r}, \omega) = \mathbf{B}_0 e^{i\mathbf{k} \cdot \mathbf{r}}. \quad (5b)$$

- (a) In the weak approximation we can replace the electric field in the expression for $\mathbf{K}(\hat{\mathbf{r}}, \omega)$ with the incident electric field when

$$\frac{k^2 \alpha}{r} < 1, \quad (6)$$

where $\alpha = \int d^3r [\chi(\mathbf{r})] \sim l^3$ is the measure of the size of the obstacle. Since we necessarily have $l \ll r$ for the far-field regions we have the requirement for the weak approximation satisfied when

$$l \ll \lambda. \quad (7)$$

That is, the dimensions of the scatterer are significantly smaller than the wavelength of the incident plane wave. This is the Rayleigh scattering approximation. In this approximation show that

$$\mathbf{K}(\theta, \phi, \omega) = -\frac{k^2}{4\pi} \chi(\mathbf{k}' - \mathbf{k}, \omega) \cdot \mathbf{E}_0, \quad (8)$$

where we identified the integral to be the Fourier transform

$$\chi(\mathbf{q}, \omega) = \int d^3r' e^{-i\mathbf{q}\cdot\mathbf{r}'} \chi(\mathbf{r}', \omega). \quad (9)$$

(b) For the case of isotropic scatterer we have

$$\chi(\mathbf{r}, \omega) = \mathbf{1}\chi(\mathbf{r}, \omega). \quad (10)$$

Thus, for an isotropic scatterer, show that we can write

$$\mathbf{K}(\theta, \phi, \omega) = \mathbf{E}_0 f(\theta, \phi, \omega) \quad (11)$$

in terms of the scattering amplitude

$$f(\theta, \phi, \omega) = -\frac{k^2}{4\pi} \chi(\mathbf{k}' - \mathbf{k}, \omega) \quad (12)$$

which involves the Fourier transform of the susceptibility. For the isotropic scatterer, show that the (Sommerfeld) boundary conditions can be expressed in the form

$$\lim_{r \rightarrow \infty} \mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{E}_0) f(\theta, \phi, \omega) \frac{e^{ikr}}{r}, \quad (13a)$$

$$\lim_{r \rightarrow \infty} c\mathbf{B}(\mathbf{r}, \omega) = c\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}} - (\hat{\mathbf{r}} \times \mathbf{E}_0) f(\theta, \phi, \omega) \frac{e^{ikr}}{r}. \quad (13b)$$

(c) A point polarizable atom is described by the susceptibility

$$\chi(\mathbf{r}, \omega) = 4\pi\alpha(\omega) \delta^{(3)}(\mathbf{r} - \mathbf{s}), \quad (14)$$

where \mathbf{s} is the position of the scatterer. Here $\alpha(\omega)$ is the polarizability of the scatterer, that is frequency dependent and has dimensions of length-cube. Show that these replacements lead to the scattering amplitude

$$\mathbf{K}(\hat{\mathbf{r}}, \omega) = -k^2 \alpha(\omega) \cdot \mathbf{E}_0 e^{i(\mathbf{k}' - \mathbf{k})\cdot\mathbf{s}}. \quad (15)$$

The scattering amplitude off a point scatterer is the basic building block in the study of scattering off a crystal that constitutes a lattice construction of point scatterers. Scattering cross section being a bilinear construction of the scattering amplitude has contributions from interference of two scatterers. For point scatterers with isotropic polarizabilities we have $\boldsymbol{\alpha}(\omega) = \mathbf{1}\alpha(\omega)$ and the scattering amplitude takes the form

$$f(\hat{\mathbf{r}}, \omega) = -k^2 \alpha(\omega) e^{i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{s}}. \quad (16)$$

2. **(20 points.)** The scattering amplitude off a scatterer of susceptibility

$$\chi(\mathbf{r}, \omega) = \frac{\varepsilon(\mathbf{r}, \omega)}{\varepsilon_0} - \mathbf{1} \quad (17)$$

is given by

$$f(\theta, \phi, \omega) = -\frac{k^2}{4\pi} \chi(\mathbf{k}' - \mathbf{k}, \omega), \quad (18)$$

where $\chi(\mathbf{q}, \omega)$ is the Fourier transform of $\chi(\mathbf{r}, \omega)$,

$$\chi(\mathbf{q}, \omega) = \int d^3r e^{i\mathbf{q} \cdot \mathbf{r}} \chi(\mathbf{r}, \omega) \quad (19)$$

and

$$k = \frac{\omega}{c}. \quad (20)$$

What is the dimension of the scattering amplitude?

3. **(20 points.)** The scattering amplitude off a scatterer of susceptibility $\chi(\mathbf{r}, \omega)$ is given by

$$f(\theta, \phi, \omega) = -\frac{k^2}{4\pi} \chi(\mathbf{k}' - \mathbf{k}, \omega), \quad (21)$$

where $\chi(\mathbf{q}, \omega)$ is the Fourier transform of $\chi(\mathbf{r}, \omega)$,

$$\chi(\mathbf{q}, \omega) = \int d^3r e^{i\mathbf{q} \cdot \mathbf{r}} \chi(\mathbf{r}, \omega). \quad (22)$$

If the scatterers are confined on a plane, say $z = 0$, then it is convenient to define polarizability per unit area $\boldsymbol{\lambda} = \boldsymbol{\alpha}/\text{Area}$, and the associated susceptibility

$$\boldsymbol{\chi}(\mathbf{r}, \omega) = 4\pi \boldsymbol{\lambda}(\mathbf{s}) \delta(z), \quad (23)$$

where the δ -function has been used to describe the assumption that the obstacles in a thin film are confined to the $z = 0$ plane here. Once the scatterers are restricted on a plane, we can choose the direction of incidence of the plane wave \mathbf{k} to be normal to the plane constituting the scatterers. That is, $\mathbf{k} \cdot \mathbf{s} = 0$, where \mathbf{s} are the positions of the point scatterers on the plane $z = 0$. Also, note that the amplitude of the incident electric field

\mathbf{E}_0 is independent of the position \mathbf{s} . Using these considerations show that the scattering amplitude, for isotropic polarizabilities, $\boldsymbol{\lambda}(\mathbf{s}) = \mathbf{1}\lambda(\mathbf{s})$, is given by

$$f(\theta, \phi, \omega) = -k^2 \int d^2s e^{ik\hat{\mathbf{r}}\cdot\mathbf{s}} \lambda(\mathbf{s}). \quad (24)$$

For a disc of radius R centered at position \mathbf{s}_0 with uniform polarizability per unit area λ complete the integrals to obtain

$$f(\theta, \phi, \omega) = -\lambda k^2 \pi R^2 2 \frac{J_1(kR \sin \theta)}{kR \sin \theta} e^{ik\hat{\mathbf{r}}\cdot\mathbf{s}_0}. \quad (25)$$

Hint: Use the integral representation of zeroth order Bessel function of the first kind

$$J_0(t) = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{it \cos \phi} \quad (26)$$

and the identity

$$\int_0^b t dt J_0(t) = b J_1(b). \quad (27)$$

Note the limiting value

$$\lim_{x \rightarrow 0} \frac{J_1(x)}{x} = \frac{1}{2}, \quad (28)$$

which guarantees a well defined value for the scattering amplitude at $\theta = 0$. We observe the interesting feature that the scattering amplitude at $\theta = 0$ is entirely given by the area of the disc.