

Homework No. 02 (Fall 2026)

PHYS 500: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2026 Aug 31, 4.30pm

1. **(20 points.)** Show that, if $\mathbf{r}$ and $\mathbf{p}$ are Hermitian, the operator construction for angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \quad (1)$$

is also Hermitian. Show that the symmetric construction

$$\mathbf{L} = \frac{1}{2} \left[(\mathbf{r} \times \mathbf{p}) + (\mathbf{r} \times \mathbf{p})^\dagger \right] \quad (2)$$

is also Hermitian. Evaluate the measure of non-commutativity of the angular momentum operator by evaluating

$$\frac{1}{2} \left[(\mathbf{r} \times \mathbf{p}) - (\mathbf{r} \times \mathbf{p})^\dagger \right]. \quad (3)$$

2. **(20 points.)** The components of the position and momentum operator, $\mathbf{r}$ and $\mathbf{p}$, respectively, satisfy the commutation relations $[r_i, p_j] = i\hbar\delta_{ij}$. Verify the following:

(a) $\mathbf{r} \times \mathbf{p} + \mathbf{p} \times \mathbf{r} = 0$.

(b) $\mathbf{r} \cdot \mathbf{p} - \mathbf{p} \cdot \mathbf{r} = 3i\hbar$.

(c) $(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{p}) - (\mathbf{b} \cdot \mathbf{p})(\mathbf{a} \cdot \mathbf{r}) = i\hbar(\mathbf{a} \cdot \mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ are numerical.

(d) $\mathbf{r} \times (\mathbf{r} \times \mathbf{p}) = \mathbf{r} \mathbf{p} \cdot \mathbf{r} - \mathbf{p} r^2 + i\hbar \mathbf{r}$.

3. **(20 points.)** Use index notation or dyadic notation to show that

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}, \quad (4a)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B}), \quad (4b)$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{A} + \mathbf{A}(\nabla \cdot \mathbf{B}) - (\nabla \cdot \mathbf{A}) \mathbf{B} - (\mathbf{A} \cdot \nabla) \mathbf{B}. \quad (4c)$$

4. **(20 points.)** Consider the dyadic construction

$$\mathbf{T} = \mathbf{E} \mathbf{B} \quad (5)$$

built using the vector fields,

$$\mathbf{E} = E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}}, \quad (6a)$$

$$\mathbf{B} = B \hat{\mathbf{y}}. \quad (6b)$$

Evaluate the following components of the dyadic:

$$\hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \quad \hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{y}} = \quad \hat{\mathbf{x}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} = \quad (7a)$$

$$\hat{\mathbf{y}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \quad \hat{\mathbf{y}} \cdot \mathbf{T} \cdot \hat{\mathbf{y}} = \quad \hat{\mathbf{y}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} = \quad (7b)$$

$$\hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{x}} = \quad \hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{y}} = \quad \hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} = \quad (7c)$$

Evaluate the scalars

$$\text{Tr}(\mathbf{T}) = T_{ii}, \quad (8a)$$

$$\text{Tr}(\mathbf{T} \cdot \mathbf{T}) = T_{ij}T_{ji}, \quad (8b)$$

$$\text{Tr}(\mathbf{T} \cdot \mathbf{T} \cdot \mathbf{T}) = T_{ij}T_{jk}T_{ki}. \quad (8c)$$

Evaluate the following vector field constructions:

$$\hat{\mathbf{x}} \cdot \mathbf{T} = \quad \hat{\mathbf{y}} \cdot \mathbf{T} = \quad \hat{\mathbf{z}} \cdot \mathbf{T} = \quad (9a)$$

$$\mathbf{T} \cdot \hat{\mathbf{x}} = \quad \mathbf{T} \cdot \hat{\mathbf{y}} = \quad \mathbf{T} \cdot \hat{\mathbf{z}} = \quad (9b)$$

$$\hat{\mathbf{x}} \times \mathbf{T} \cdot \hat{\mathbf{x}} = \quad \hat{\mathbf{y}} \times \mathbf{T} \cdot \hat{\mathbf{x}} = \quad \hat{\mathbf{z}} \times \mathbf{T} \cdot \hat{\mathbf{x}} = \quad (9c)$$

$$\hat{\mathbf{x}} \times \mathbf{T} \cdot \hat{\mathbf{y}} = \quad \hat{\mathbf{y}} \times \mathbf{T} \cdot \hat{\mathbf{y}} = \quad \hat{\mathbf{z}} \times \mathbf{T} \cdot \hat{\mathbf{y}} = \quad (9d)$$

$$\hat{\mathbf{x}} \times \mathbf{T} \cdot \hat{\mathbf{z}} = \quad \hat{\mathbf{y}} \times \mathbf{T} \cdot \hat{\mathbf{z}} = \quad \hat{\mathbf{z}} \times \mathbf{T} \cdot \hat{\mathbf{z}} = \quad (9e)$$

$$\hat{\mathbf{x}} \cdot \mathbf{T} \times \hat{\mathbf{x}} = \quad \hat{\mathbf{x}} \cdot \mathbf{T} \times \hat{\mathbf{y}} = \quad \hat{\mathbf{x}} \cdot \mathbf{T} \times \hat{\mathbf{z}} = \quad (9f)$$

$$\hat{\mathbf{y}} \cdot \mathbf{T} \times \hat{\mathbf{x}} = \quad \hat{\mathbf{y}} \cdot \mathbf{T} \times \hat{\mathbf{y}} = \quad \hat{\mathbf{y}} \cdot \mathbf{T} \times \hat{\mathbf{z}} = \quad (9g)$$

$$\hat{\mathbf{z}} \cdot \mathbf{T} \times \hat{\mathbf{x}} = \quad \hat{\mathbf{z}} \cdot \mathbf{T} \times \hat{\mathbf{y}} = \quad \hat{\mathbf{z}} \cdot \mathbf{T} \times \hat{\mathbf{z}} = \quad (9h)$$

5. **(20 points.)** Verify the following identities:

$$\nabla r = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}, \quad (10a)$$

$$\nabla \mathbf{r} = \mathbf{1}. \quad (10b)$$

Further, show that

$$\nabla \cdot \mathbf{r} = 3, \quad (11a)$$

$$\nabla \times \mathbf{r} = \mathbf{0}. \quad (11b)$$

Here r is the magnitude of the position vector $\mathbf{r}$, and $\hat{\mathbf{r}}$ is the unit vector pointing in the direction of $\mathbf{r}$.

6. **(20 points.)** Evaluate the left hand side of the equation

$$\nabla(\mathbf{r} \cdot \mathbf{p}) = a \mathbf{p} + b \mathbf{r}, \quad (12)$$

where $\mathbf{p}$ is a constant vector. Thus, find a and b .

7. **(20 points.)** Evaluate

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right), \quad (13)$$

everywhere in space, including $\mathbf{r} = 0$.

Hint: Check your answer for consistency by using divergence theorem.

8. **(20 points.)** Evaluate

$$\nabla \cdot \left(\frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right), \quad (14)$$

where $\mathbf{p}$ is a constant vector.

9. **(20 points.)** Consider the distribution

$$\delta(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}. \quad (15)$$

Show that

$$\delta(x) \begin{cases} \rightarrow \infty, & \text{if } x = 0, \\ \rightarrow 0, & \text{if } x \neq 0. \end{cases} \quad (16)$$

Further, show that

$$\int_{-\infty}^{\infty} dx \delta(x) = 1. \quad (17)$$

Plot $\delta(x)$ before taking the limit $\varepsilon \rightarrow 0$ and identify ε in the plot.

10. **(10 points.)** A uniformly charged infinitely thin disc of radius R and total charge Q is placed on the x - y plane such that the normal vector is along the z axis and the center of the disc at the origin. Write down the charge density of the disc in terms of δ -function(s). Integrate over the charge density and verify that it returns the total charge on the disc.
11. **(10 points.)** An (idealized) infinitely long wire, (on the z -axis with infinitesimally small cross sectional area,) carrying a current I can be mathematically represented by the current density

$$\mathbf{J}(\mathbf{x}) = \hat{\mathbf{z}} I \delta(x) \delta(y). \quad (18)$$

A similar idealized wire forms a circular loop and is placed on the xy -plane with the center of the circular loop at the origin. Write down the current density of the circular loop carrying current I .