

Homework No. 03 (Fall 2026)

PHYS 500: MATHEMATICAL METHODS

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Due date: Friday, 2026 Sep 11, 4.30pm

0. Problem 1 is for practice. Problems 2 and 3 are for submission.

1. (**Example.**) Let $\mathbf{r}$ represent the position vector, x^i the components of the position vector in rectangular coordinates, and u^i the components of the position vector in cylindrical polar coordinates. In particular, we have

$$x^1 = x = \rho \cos \phi, \quad u^1 = \rho = \sqrt{x^2 + y^2}, \quad (1a)$$

$$x^2 = y = \rho \sin \phi, \quad u^2 = \phi = \tan^{-1} \frac{y}{x}, \quad (1b)$$

$$x^3 = z = z, \quad u^3 = z = z. \quad (1c)$$

Let us define the unit vectors

$$\hat{\boldsymbol{\rho}} = \cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}} + 0 \hat{\mathbf{k}}, \quad (2a)$$

$$\hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}} + 0 \hat{\mathbf{k}}, \quad (2b)$$

$$\hat{\mathbf{z}} = 0 \hat{\mathbf{i}} + 0 \hat{\mathbf{j}} + \hat{\mathbf{k}}, \quad (2c)$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are basis vectors in rectangular coordinate system. We will also use the notation $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}^3 = \hat{\mathbf{x}}_3$, to represent these vectors.

(a) Basis vectors:

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i}. \quad (3)$$

Show that

$$\mathbf{e}_1 = \hat{\boldsymbol{\rho}}, \quad \mathbf{e}_2 = \rho \hat{\boldsymbol{\phi}}, \quad \mathbf{e}_3 = \hat{\mathbf{z}}. \quad (4)$$

(b) Reciprocal basis vectors:

$$\mathbf{e}^i = \nabla u^i. \quad (5)$$

Show that

$$\mathbf{e}^1 = \hat{\boldsymbol{\rho}}, \quad \mathbf{e}^2 = \frac{\hat{\boldsymbol{\phi}}}{\rho}, \quad \mathbf{e}^3 = \hat{\mathbf{z}}. \quad (6)$$

(c) Orthonormality: Show that

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta_j^i. \quad (7)$$

(d) Metric tensor: A line element is defined as

$$d\mathbf{r} = dx^i \hat{\mathbf{x}}_i = du^i \mathbf{e}_i. \quad (8)$$

Show that

$$d\mathbf{r} \cdot d\mathbf{r} = du^i du^j g_{ij}, \quad (9)$$

where the metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \quad (10)$$

Evaluate all the components of g_{ij} .

(e) Completeness relation: Starting from

$$\nabla \mathbf{r} = \mathbf{1} \quad (11)$$

derive the completeness relation

$$\mathbf{e}^i \mathbf{e}_i = \mathbf{1}. \quad (12)$$

Express the completeness relation in cylindrical polar coordinates in terms of $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$.

2. (20 points.) Transformation of basis vectors.

(a) Let us consider a set of basis vectors $\mathbf{e}_i$, where $i = 1, 2, 3$, and the associated reciprocal basis vectors $\mathbf{e}^i$ that satisfy the completeness relation

$$\mathbf{1} = \mathbf{e}^i \mathbf{e}_i = \mathbf{e}^1 \mathbf{e}_1 + \mathbf{e}^2 \mathbf{e}_2 + \mathbf{e}^3 \mathbf{e}_3. \quad (13)$$

It is a complete set because an arbitrary vector $\mathbf{A}$ can be expressed in terms of its projections along the basis vectors in the following way,

$$\mathbf{A} = \mathbf{A} \cdot \mathbf{1} = \mathbf{A} \cdot (\mathbf{e}^i \mathbf{e}_i) = (\mathbf{A} \cdot \mathbf{e}^i) \mathbf{e}_i = A^i \mathbf{e}_i, \quad (14)$$

where we recognized and defined the projections of vector $\mathbf{A}$ along the direction of basis vectors as the components

$$A^i = (\mathbf{A} \cdot \mathbf{e}^i). \quad (15)$$

Similarly, multiplying by the identity dyadic on the left gives

$$\mathbf{A} = \mathbf{1} \cdot \mathbf{A} = (\mathbf{e}^i \mathbf{e}_i) \cdot \mathbf{A} = \mathbf{e}^i (\mathbf{e}_i \cdot \mathbf{A}) = \mathbf{e}^i A_i, \quad (16)$$

where now the projections of vector $\mathbf{A}$ in the direction of the reciprocal basis vectors are the components

$$A_i = (\mathbf{e}_i \cdot \mathbf{A}). \quad (17)$$

For consistency we require the equality

$$(\mathbf{A} \cdot \mathbf{e}^i) \mathbf{e}_i = \mathbf{e}^i (\mathbf{e}_i \cdot \mathbf{A}). \quad (18)$$

Thus, derive

$$A^j = g^{ji} A_i, \quad (19a)$$

$$A_j = g_{ji} A^i, \quad (19b)$$

where the metric tensors are defined as where the metric tensors are defined as

$$g^{ji} = \mathbf{e}^j \cdot \mathbf{e}^i, \quad (20a)$$

$$g_{ji} = \mathbf{e}_j \cdot \mathbf{e}_i. \quad (20b)$$

- (b) For another set of basis vectors $\mathbf{g}_i$ and the associated reciprocal basis vectors $\mathbf{g}^i$ that satisfy the completeness relation

$$\mathbf{1} = \mathbf{g}^i \mathbf{g}_i \quad (21)$$

we can write

$$\mathbf{A} = (\mathbf{A} \cdot \mathbf{g}^i) \mathbf{g}_i = \bar{A}^i \mathbf{g}_i, \quad (22)$$

where the components $\bar{A}^i$ are in general different from A^i , and

$$\mathbf{A} = \mathbf{g}^i (\mathbf{g}_i \cdot \mathbf{A}) = \mathbf{g}^i \bar{A}_i. \quad (23)$$

For consistency we require the equality

$$(\mathbf{A} \cdot \mathbf{g}^i) \mathbf{g}_i = \mathbf{g}^i (\mathbf{g}_i \cdot \mathbf{A}). \quad (24)$$

Thus, derive

$$\bar{A}^j = \bar{g}^{ji} \bar{A}_i, \quad (25a)$$

$$\bar{A}_j = \bar{g}_{ji} \bar{A}^i, \quad (25b)$$

where the metric tensors are defined as

$$\bar{g}^{ji} = \mathbf{g}^j \cdot \mathbf{g}^i, \quad (26a)$$

$$\bar{g}_{ji} = \mathbf{g}_j \cdot \mathbf{g}_i. \quad (26b)$$

- (c) For consistency between the two independent basis vector representations we require the equality

$$(\mathbf{A} \cdot \mathbf{e}^i) \mathbf{e}_i = (\mathbf{A} \cdot \mathbf{g}^i) \mathbf{g}_i. \quad (27)$$

Taking the dot product on the right with $\mathbf{e}^j$ and using orthogonality relation

$$\mathbf{e}_i \cdot \mathbf{e}^i = \delta_j^i \quad (28)$$

we obtain

$$(\mathbf{A} \cdot \mathbf{e}^j) = (\mathbf{A} \cdot \mathbf{g}^i) [\mathbf{g}_i \cdot \mathbf{e}^j]. \quad (29)$$

Similarly, taking the dot product on the right with $\mathbf{g}^j$ we obtain

$$(\mathbf{A} \cdot \mathbf{e}^i) [\mathbf{e}_i \cdot \mathbf{g}^j] = (\mathbf{A} \cdot \mathbf{g}^j). \quad (30)$$

In terms of the transformation matrices connecting the two basis vectors,

$$S_i^j = [\mathbf{g}_i \cdot \mathbf{e}^j] \quad (31)$$

and

$$R_i^j = [\mathbf{e}_i \cdot \mathbf{g}^j], \quad (32)$$

we can derive the transformation of the components

$$A^j = \bar{A}^i S_i^j, \quad (33a)$$

$$A^i R_i^j = \bar{A}^j. \quad (33b)$$

Further, derive

$$A_j = R_j^i \bar{A}_i, \quad (34a)$$

$$S_j^i A_i = \bar{A}^j. \quad (34b)$$

Show that

$$R_i^j S_j^k = [\mathbf{e}_i \cdot \mathbf{g}^j] [\mathbf{g}_j \cdot \mathbf{e}^k] = \mathbf{e}_i \cdot (\mathbf{g}^j \mathbf{g}_j) \cdot \mathbf{e}^k = \mathbf{e}_i \cdot \mathbf{1} \cdot \mathbf{e}^k = \delta_i^k. \quad (35)$$

- (d) i. Find the transformation matrices S and T between cylindrical polar coordinates and rectangular coordinates. Verify that $ST = 1$.
 ii. Find the transformation matrices S and T between cylindrical polar coordinates and spherical polar coordinates. Verify that $ST = 1$.
3. **(20 points.)** Let $\mathbf{r}$ represent a position vector in three dimensional space. Let x^i be the components of the position vector in rectangular coordinates, which can be interpreted as surfaces of constant x^i . Let us coordinatize the space using the planes, labeled using β ,

$$y = mx + \beta \quad (36)$$

where m is fixed, instead of planes with constant y . The other two sets of planes of constant x and constant z are the same. See Fig. 1. Let u^i be the components of the position vector in this new coordinatization of space. In particular, we have

$$x^1 = x = \alpha, \quad u^1 = \alpha = x, \quad (37a)$$

$$x^2 = y = mx + \beta, \quad u^2 = \beta = y - mx, \quad (37b)$$

$$x^3 = z = \gamma, \quad u^3 = \gamma = z. \quad (37c)$$

The basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ in rectangular coordinate system will be represented as $\hat{\mathbf{i}} = \hat{\mathbf{x}}^1 = \hat{\mathbf{x}}_1$, $\hat{\mathbf{j}} = \hat{\mathbf{x}}^2 = \hat{\mathbf{x}}_2$, $\hat{\mathbf{k}} = \hat{\mathbf{x}}^3 = \hat{\mathbf{x}}_3$, if necessary.

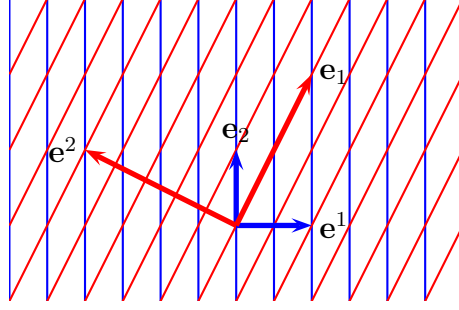


Figure 1: Basis vectors $\mathbf{e}_i$ and reciprocal basis vectors $\mathbf{e}^i$.

(a) Basis vectors:

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial u^i}. \quad (38)$$

Show that

$$\mathbf{e}_1 = \hat{\mathbf{i}} + m\hat{\mathbf{j}}, \quad \mathbf{e}_2 = \hat{\mathbf{j}}, \quad \mathbf{e}_3 = \hat{\mathbf{k}}. \quad (39)$$

(b) Reciprocal basis vectors:

$$\mathbf{e}^i = \nabla u^i. \quad (40)$$

Show that

$$\mathbf{e}^1 = \hat{\mathbf{i}}, \quad \mathbf{e}^2 = -m\hat{\mathbf{i}} + \hat{\mathbf{j}}, \quad \mathbf{e}^3 = \hat{\mathbf{k}}. \quad (41)$$

Verify the relations

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{(\mathbf{e}_2 \times \mathbf{e}_3) \cdot \mathbf{e}_1}, \quad \mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{(\mathbf{e}_3 \times \mathbf{e}_1) \cdot \mathbf{e}_2}, \quad \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3}. \quad (42)$$

(c) Orthonormality: Show that

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta_j^i. \quad (43)$$

That is,

$$\mathbf{e}^1 \cdot \mathbf{e}_1 = 1, \quad \mathbf{e}^1 \cdot \mathbf{e}_2 = 0, \quad \mathbf{e}^1 \cdot \mathbf{e}_3 = 0, \quad (44a)$$

$$\mathbf{e}^2 \cdot \mathbf{e}_1 = 0, \quad \mathbf{e}^2 \cdot \mathbf{e}_2 = 1, \quad \mathbf{e}^2 \cdot \mathbf{e}_3 = 0, \quad (44b)$$

$$\mathbf{e}^3 \cdot \mathbf{e}_1 = 0, \quad \mathbf{e}^3 \cdot \mathbf{e}_2 = 0, \quad \mathbf{e}^3 \cdot \mathbf{e}_3 = 1. \quad (44c)$$

(d) Metric tensor: The metric tensor g_{ij} is defined as

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \quad (45)$$

Evaluate all the components of g_{ij} . That is,

$$g_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1 + m^2, \quad g_{12} = \mathbf{e}_1 \cdot \mathbf{e}_2 = m, \quad g_{13} = \mathbf{e}_1 \cdot \mathbf{e}_3 = 0, \quad (46a)$$

$$g_{21} = \mathbf{e}_2 \cdot \mathbf{e}_1 = m, \quad g_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1, \quad g_{23} = \mathbf{e}_2 \cdot \mathbf{e}_3 = 0, \quad (46b)$$

$$g_{31} = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0, \quad g_{32} = \mathbf{e}_3 \cdot \mathbf{e}_2 = 0, \quad g_{33} = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1. \quad (46c)$$

Similarly evaluate the components of

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j. \quad (47)$$

That is,

$$g^{11} = \mathbf{e}^1 \cdot \mathbf{e}^1 = 1, \quad g^{12} = \mathbf{e}^1 \cdot \mathbf{e}^2 = -m, \quad g^{13} = \mathbf{e}^1 \cdot \mathbf{e}^3 = 0, \quad (48a)$$

$$g^{21} = \mathbf{e}^2 \cdot \mathbf{e}^1 = -m, \quad g^{22} = \mathbf{e}^2 \cdot \mathbf{e}^2 = 1 + m^2, \quad g^{23} = \mathbf{e}^2 \cdot \mathbf{e}^3 = 0, \quad (48b)$$

$$g^{31} = \mathbf{e}^3 \cdot \mathbf{e}^1 = 0, \quad g^{32} = \mathbf{e}^3 \cdot \mathbf{e}^2 = 0, \quad g^{33} = \mathbf{e}^3 \cdot \mathbf{e}^3 = 1. \quad (48c)$$

Verify that $g^{ij}g_{jk} = \delta_k^i$.

(e) Completeness relation: Verify the completeness relation

$$\mathbf{e}^i \mathbf{e}_i = \mathbf{1} \quad (49)$$

by evaluating

$$\mathbf{e}^1 \mathbf{e}_1 + \mathbf{e}^2 \mathbf{e}_2 + \mathbf{e}^3 \mathbf{e}_3. \quad (50)$$

(f) Given a vector

$$\mathbf{A} = a \hat{\mathbf{i}} + b \hat{\mathbf{j}} + c \hat{\mathbf{k}} \quad (51)$$

in rectangular coordinates, find the components of the vector $\mathbf{A}$ in the basis of $\mathbf{e}_i$. That is, find the components A^i in

$$\mathbf{A} = A^1 \mathbf{e}_1 + A^2 \mathbf{e}_2 + A^3 \mathbf{e}_3. \quad (52)$$