

Homework No. 04 (Fall 2026)

PHYS 500: MATHEMATICAL METHODS

School of Physics and Applied Physics, Southern Illinois University–Carbondale

Due date: Monday, 2026 Sep 21, 4.30pm

1. **(20 points.)** For a given complex number z , say

$$z = \sqrt{2} e^{i\frac{\pi}{3}}, \quad (1)$$

evaluate

$$z^2, z^3, z^4, z^5, z^6, z^7, z^8, z^9, z^{10}. \quad (2)$$

Mark all of them on the complex plane. Decipher the pattern.

2. **(20 points.)** Evaluate

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2} \right)^{23}. \quad (3)$$

Mark the resulting number on the complex plane.

3. **(20 points.)** Find the fifth roots of unity by solving the equation

$$z^5 = 1. \quad (4)$$

Mark the points corresponding to the five roots on the complex plane. Find the five roots of the equation

$$z^5 = -1. \quad (5)$$

Mark the roots on the complex plane. Next, find the roots of the equation

$$z^5 = i \quad (6)$$

and mark the roots on the complex plane. Repeat the exercise for $z^5 = -i$. How do these roots match with the fifth roots of unity? Recognize the pattern.

4. **(20 points.)** Recall that analytic functions satisfy the Cauchy-Riemann equations. That is, the real and imaginary parts of an analytic function

$$f(x + iy) = u(x, y) + iv(x, y) \quad (7)$$

satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad (8a)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \quad (8b)$$

Given $f(z)$ and $g(z)$ are analytic functions in a region, then show that $f(g(z))$ satisfies the Cauchy-Riemann equations there.

Hint: Let $g = u + iv$ and $f = U + iV$. Thus, we can write

$$f(g(z)) = U(u(x, y), v(x, y)) + iV(u(x, y), v(x, y)). \quad (9)$$

5. **(20 points.)** Analytic functions are significantly constrained, in that they have to satisfy the Cauchy-Riemann conditions. These conditions are necessary (but not sufficient) for a function of a complex variable to be analytic (differentiable). Check if the following functions satisfy the Cauchy-Riemann conditions. If $f(z)$ is analytic for all z , then report the derivative as a function of z . Otherwise, determine the points, or regions, in the z plane where the function is not analytic.

$$f(z) = z^3, \quad (10a)$$

$$f(z) = |z|^2, \quad (10b)$$

$$f(z) = e^{iz}, \quad (10c)$$

$$f(z) = \ln z, \quad (10d)$$

$$f(z) = e^z + e^{iz}. \quad (10e)$$

Mathematica allows visualization of these functions using the command `ComplexContourPlot`. [Check out <https://www.wolframcloud.com/>.]

```
f[z_] = z^3;
ComplexContourPlot[ReIm[f[z]], {z, -3-3 I, 3+3 I}]
```

The above two-line code in Mathematica plots the real and imaginary surfaces associated with the analytic function $f(z) = z^3$ between the coordinate points $(-3, -3)$ and $(3, 3)$.

Use `ComplexContourPlot` in Mathematica to visualize these functions.