Homework No. 03 (Fall 2026)

PHYS 520A: ELECTROMAGNETIC THEORY I

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Due date: Tuesday, 2026 Sep 15, 4.30pm

1. (**20 points.**) The following representation for a δ-function,
$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}, \tag{1}$$
is used in the context of Fourier transformations. A δ-function is characterized by the properties that
$$\delta(x) = 0 \qquad \text{if} \qquad x \neq 0, \tag{2}$$
and
$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{3}$$
The above two properties are incompatible because in standard calculus the value of a definite integral is independent of the integrand at a single isolated point. Thus, after $x = 0$ is removed from the span of the integral, how can a sum of zeros add up to a non-zero value of 1. However, the above representation for a δ-function has an interpretation in standard calculus, as a regularization limit, after a damping factor has been introduced. Let
$$I_\varepsilon(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx-|k|\varepsilon}, \qquad \varepsilon > 0. \tag{4}$$
Evaluate the integral in Eq. (4) to show that
$$I_\varepsilon(x) = \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}. \tag{5}$$
Thus, as a regularization limit, the representation is
$$\delta(x) = \lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx-|k|\varepsilon} \tag{6a}$$
$$= \lim_{\varepsilon\to 0+} \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}. \tag{6b}$$
Show that both properties of a δ-function in Eqs. (2) and (3) are satisfied by the representation in Eq. (6).

2. (**20 points.**) The integral
$$\int_0^{\infty} dx \left(\frac{\sin x}{x}\right) = \frac{\pi}{2} \tag{7}$$

can be evaluated by continuing to the complex plane. Otherwise, to evaluate the integral, construct

$$I(a) = \int_0^\infty \frac{dx}{x}\, e^{-ax} \sin x \tag{8}$$

and learn that

$$-I'(a) = \int_0^\infty dx\, e^{-ax} \sin x. \tag{9}$$

Integrating by parts twice deduce

$$-I'(a) = \frac{1}{1+a^2} \tag{10}$$

and integrate to conclude

$$I(a) = \frac{\pi}{2} - \tan^{-1} a. \tag{11}$$

3. (**20 points.**) The Green function for the negative Laplacian operator is defined using Green's function equation,

$$-\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta^{(3)}(\mathbf{r} - \mathbf{r}'). \tag{12}$$

(a) The charge density for a point charge q is

$$\rho(\mathbf{r}) = q\delta^{(3)}(\mathbf{r} - \mathbf{r}'), \tag{13}$$

where $\mathbf{r}'$ is the position of the charge. Verify that the integral of the charge density over all space,

$$\int d^3r\, \rho(\mathbf{r}) = q, \tag{14}$$

yields the charge q. The Poisson equation for the electric potential due to the point charge is

$$-\nabla^2 \varepsilon_0 \phi(\mathbf{r}) = q\delta^{(3)}(\mathbf{r} - \mathbf{r}'). \tag{15}$$

Thus, the Green function is the electric potential due to a point charge, upto a factor q/ε_0,

$$\varepsilon_0 \phi(\mathbf{r}) = qG(\mathbf{r}, \mathbf{r}'). \tag{16}$$

(b) The Green function has to have the functional dependence of the form $\mathbf{r} - \mathbf{r}'$ because it should have translation symmetry in the space constituting of a single point charge. Thus,

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r} - \mathbf{r}'). \tag{17}$$

Using Fourier transform of the Green function

$$G(\mathbf{r} - \mathbf{r}') = \int \frac{d^{(3)}k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} G(\mathbf{k}) \tag{18}$$

and the Fourier transform of δ-function

$$\delta^{(3)}(\mathbf{r} - \mathbf{r}') = \int \frac{d^{(3)}k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} 1 \tag{19}$$

show that

$$G(\mathbf{k}) = \frac{1}{k^2}. \tag{20}$$

(c) Thus, show that the Green function is

$$G(\mathbf{r}-\mathbf{r}') = \int \frac{d^{(3)}k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2}. \tag{21}$$

Using spherical coordinates for $\mathbf{k}$ and choosing $\mathbf{r}-\mathbf{r}'$ along the z-axis show that

$$\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') = kR\cos\theta, \quad R = |\mathbf{r}-\mathbf{r}'|, \quad d^3k = k^2 \sin\theta d\theta d\phi. \tag{22}$$

Then, complete the θ and ϕ integrals to derive

$$G(\mathbf{r}-\mathbf{r}') = \frac{1}{4\pi R}\frac{2}{\pi}\int_0^\infty dx \left(\frac{\sin x}{x}\right). \tag{23}$$

Evaluate the integral to derive

$$G(\mathbf{r}-\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}, \tag{24}$$

which is the Coulomb potential due to a point charge, upto a factor q/ε_0.

4. (**20 points.**) The superposition principle for the electric potential in electrostatics states that the electric potential at a given point in space is the sum of the electric potentials produced by all point charges. To derive the superposition principle we start from the Poisson equation corresponding to a charge density $\rho(\mathbf{r})$,

$$-\nabla^2 \varepsilon_0 \phi(\mathbf{r}) = \rho(\mathbf{r}). \tag{25}$$

The Green function equation is

$$-\nabla^2 G(\mathbf{r}-\mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{26}$$

(a) Multiply the Poisson equation by the Green function, after replacing $\mathbf{r}\to\mathbf{r}'$, and integrate over all space, to obtain

$$-\int d^3r' G(\mathbf{r}-\mathbf{r}')\nabla'^2 \varepsilon_0\phi(\mathbf{r}') = \int d^3r' G(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}'). \tag{27}$$

Then, integrate by parts twice to show that

$$\begin{aligned} -\int d^3r' \left\{\nabla'^2 G(\mathbf{r}-\mathbf{r}')\right\}\varepsilon_0\phi(\mathbf{r}') &= \int d^3r' G(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') \\ &+ \int d^3r' \boldsymbol{\nabla}'\cdot\Big[G(\mathbf{r}-\mathbf{r}')\boldsymbol{\nabla}'\varepsilon_0\phi(\mathbf{r}') - \{\boldsymbol{\nabla}' G(\mathbf{r}-\mathbf{r}')\}\varepsilon_0\phi(\mathbf{r}')\Big]. \end{aligned} \tag{28}$$

(b) Use the Green function equation in the term on the left and the divergence theorem on the two terms expresse as a total divergence to write

$$\varepsilon_0\phi(\mathbf{r}) = \int d^3r' G(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') + \oint d\mathbf{a}' \cdot \Big[G(\mathbf{r}-\mathbf{r}')\boldsymbol{\nabla}'\varepsilon_0\phi(\mathbf{r}') - \{\boldsymbol{\nabla}' G(\mathbf{r}-\mathbf{r}')\}\,\varepsilon_0\phi(\mathbf{r}')\Big]. \tag{29}$$

Using the expression for Green function

$$G(\mathbf{r}-\mathbf{r}') = \frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{30}$$

and introducing the electric field in the first surface term

$$\mathbf{E}(\mathbf{r}') = -\boldsymbol{\nabla}'\phi(\mathbf{r}') \tag{31}$$

show that

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{4\pi}\oint d\mathbf{a}' \cdot \left[\frac{\mathbf{E}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\phi(\mathbf{r}') \right]. \tag{32}$$

(c) Thus, show that for boundary conditions when the electric potential is zero for $r \to \infty$ and the electric field converges faster than $1/r$ we have

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{33}$$

which is the statement of superposition principle.